# *Compactness and Non-compactness*

**MTH 849: PDE2** *(Spring 2022)*

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## Contents



## §1: Introduction and motivation

Our starting point of these notes are two observations:

1. By the Rellich-Kondrachov Theorem, we know that if $\Omega \subset \mathbb{R}^d$ is a bounded domain with $C^1$ boundary, then we have:

   **Theorem (RK).** Given $p, q \in [1, \infty)$ satisfying $\frac{1}{q} > \frac{1}{p} - \frac{1}{d}$, then the embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ is compact.

2. The special case of the above compactness theorem, where $p = q = 2$, is used frequently in the study of *weak solutions* of elliptic PDEs. Specifically, noting that the solution operator $L_\gamma^{-1}$ for a problem provides a continuous mapping $L^2 \to H_0^1$, this allowed us to apply the theory of compact operators on $L^2$ to develop Fredholm theory. And as we saw in class, a large part of the application of compactness is the extraction of convergent subsequences.

Our goal is to further explore these topics. The main question we hope to address is the following:

> **Question.** In class (and in your homework), we have already shown that both the boundedness of the domain $\Omega$ and the strictness of the inequality $q < \frac{dp}{d-p}$ is required for the compactness of the $W^{1,p} \hookrightarrow L^q$ embedding. (We will recall these below.) Is there a way to overcome these two issues and get some remnant of compactness?

Specifically, we want to examine whether some degree of compactness remains in the $W^{1,2}(\mathbb{R}^d) \hookrightarrow L^q(\mathbb{R}^d)$ embedding, for $q \in [2, \frac{2d}{d-2})$; and whether some degree of compactness remains in the $W^{1,2} \hookrightarrow L^q$ embedding for $q = \frac{2d}{d-2}$.

It turns out that these questions can be answered by the powerful machinery of *concentration compactness*, which was originally formulated by P.L. Lions[1]. The method has since then gained traction as a powerful tool in the study of nonlinear partial differential equations. The functional analytic foundations of the method is discussed in detail in the book by Tintarev and Fieseler[2]. In these notes we will present a simple *example* of the theory, focused on the space $H^1(\Omega)$.

**§1.1 Model problem.—** To make our discussion more concrete and focussed, let's think about the following problem.

**Model Problem**. Let $\Omega$ be a domain, and let $q$ be an admissible exponent such that the Sobolev embedding $H^1(\Omega) \hookrightarrow L^q(\Omega)$ holds. Recall that this states: there exists $C > 0$ such that for every $u \in H^1(\Omega)$, we have

$$\|u\|_{L^q(\Omega)} \le C\|u\|_{H^1(\Omega)}.$$

Let $C_*$ denote the infimum of all $C$ for which the Sobolev inequality holds; this is a well-defined positive number. We wish to ask: *does there exist a non-trivial $u_* \in H^1(\Omega)$ such that $\|u_*\|_{L^q} = C_*\|u_*\|_{H^1(\Omega)}$?*

We call such an $u_*$ an *optimizer* for the Sobolev embedding.

**Theorem 1.1.** *Let $d \ge 3$ and let $\Omega \subseteq \mathbb{R}^d$ be bounded, then there is an optimizer for the $H^1_0(\Omega) \hookrightarrow L^q(\Omega)$ embedding when $q \in [2, \frac{2d}{d-2})$. Additionally, if $\Omega$ has $C^1$ boundary (so it is an extension domain), then there is an optimizer for the $H^1(\Omega) \hookrightarrow L^q(\Omega)$ embedding.*

*Proof.* The proof of the case where $\Omega$ is an extension domain is mostly the same as the first case, so we will just do the first case for brevity.

We have the following characterization of $C_*$. Note that if $C$ is such that the Sobolev embedding holds, then we have

$$\frac{1}{C} \le \frac{\|u\|_{H^1_0}}{\|u\|_{L^q}}$$

for all $u \neq 0$. In other words, $\frac{1}{C}$ is a lower bound for the set

$$\Big\{\|u\|_{H^1} : u \in H^1_0(\Omega), \|u\|_{L^q} = 1\Big\}.$$

By definition, $\frac{1}{C_*}$ is the greatest such lower bound, and hence

$$\frac{1}{C_*} = \inf\Big\{\|u\|_{H^1} : u \in H^1_0(\Omega), \|u\|_{L^q} = 1\Big\}.$$

[1] The method actually predates Lions, and have seen several uses by people solving individual problems. Lions is generally credited with recognizing the common thread behind those techniques and formulating it as an abstract principle. See his series of papers, all titled "The concentration compactness principle in the calculus of variations" in 1984–1985.

[2] K. Tintarev and K.-H. Fieseler, *Concentration Compactness: Functional-Analytic Grounds and Applications*, Imperial College Press (2007).

By the definition of infimum, there must then exist a sequence $\{u_k\}$ of $H_0^1$ functions, with $\|u_k\|_{L^q} = 1$ for every $k$, such that $\lim \|u_k\|_{H^1} = \frac{1}{C_*}$. This convergence implies that $\{u_k\}$ is bounded in $H_0^1$.

By the Banach-Alaoglu Theorem, norm bounded subsets in a Hilbert space is weakly pre-compact, and hence there exists a subsequence (which we will by abuse of notation also call $\{u_k\}$) that weakly converges to an element $u_* \in H_0^1(\Omega)$. Notice that

$$0 \leq \liminf \|u_k - u_*\|^2 = \liminf \Big( \|u_k\|^2 - 2\langle u_k, u_* \rangle + \|u_*\|^2 \Big).$$

Using that weak convergence guarantees

$$\lim \langle u_k, u_* \rangle = \|u_*\|^2$$

we find as a conclusion, the well-known fact that for weakly convergent sequences in a Hilbert space

$$\|u_*\|_{H_0^1}^2 \leq \liminf \|u_k\|_{H_0^1}^2. \tag{1.2}$$

As by construction $\{u_k\}$ is a minimizing sequence, we find

$$\|u_*\|_{H_0^1} \leq \frac{1}{C_*}.$$

By the Rellich-Kondrachov Theorem, some further subsequence (which again we still write as $\{u_k\}$ will converge strongly in $L^q$, call the limiting function $\tilde{u}$. Since $q \geq 2$ and $\Omega$ is bounded, we must have that $u_k \to \tilde{u}$ strongly also in $L^2$. Then it follows that $u_k$ converges weakly to $\tilde{u}$ in $L^2$ also. On the other hand, given any $g \in L^2$, by the Riesz representation theorem there exists $\tilde{g} \in H_0^1$ such that $\langle v, \tilde{g} \rangle_{H_0^1} = \int_\Omega vg$. Thus we see that as $u_k$ converges weakly to $u_*$ in $H_0^1$, we also have $u_k$ converging weakly to $u_*$ in $L^2$. But the uniqueness of weak limits then states that $u_* = \tilde{u}$.

The strong convergence in $L^q$ shows therefore that $\|u_*\|_{L^q} = 1$. By definition of $C^*$ we must have then

$$\|u_*\|_{H_0^1} \geq \frac{1}{C_*}.$$

Using our earlier conclusion, we combine to find that $\|u_*\|_{H_0^1} = \frac{1}{C_*}$ and hence is the optimizer that we seek. □

**Remark 1.3** (Connection to PDEs)**.** One may wonder: what does the model problem have to do with partial differential equations? As discussed in the proof above, the optimizer $u_*$ is the minimizer for the constrained problem "minimize $\|u\|_{H^1}$ for $u \in H_0^1(\Omega)$ under the constraint that $\|u\|_{L^q} = 1$." If this problem seems familiar to you, it is because we've in this class dealt with a similar problem before.

Previously we studied the *eigenvalue problem* for elliptic operators. And using Rayleigh's formula, we were about to formulate the problem of finding the principal eigenvalue and eigenfunction of the Laplacian to a similar minimization problem. More precisely, Rayleign's formula give the principal eigenvalue $\lambda_1$ and principal eigenfunction $u_1$ of the operator $-\Delta$ on a bounded domain $\Omega$ by casting it as the minimization of $\|\nabla u\|_{L^2}$ under the constraint $\|u\|_{L^2} = 1$. Recall further that this can be interpreted as finding a weak solution to the partial differential equation $-\Delta u = \lambda u$ for $u \in H_0^1$.

It turns out that the model problem above is similarly related to a *nonlinear* PDE. Knowing that the minimizer $u_*$ exists, by Fermat's principle together with Lagrange multipliers, we know that there exists a $\lambda$ such that for any $v \in H^1_0$,

$$\lim_{s\to 0} \frac{1}{s}\left(\|u_* + sv\|^2_{H^1} - \|u^*\|^2_{H^1} + \lambda \|u_* + sv\|^q_{L^q} - \lambda \|u_*\|^q_{L^q}\right) = 0.$$

(Here $\lambda$ is the Lagrange multiplier, and we write the constraint as $\|u\|^q_{L^q} = 1$, and the quantity to minimize as $\|u\|^2_{H^1}$.) Using that

$$\|u_* + sv\|^2_{H^1} = \|u_*\|^2_{H^1} + s^2\|v\|^2_{H^1} + 2s \int \nabla u_* \cdot \nabla v + uv \; dx$$

and

$$\|u_* + sv\|^q_{L^q} = \|u_*\|^q_{L^q} + qs \int |u_*|^{q-2} u_* v + O(s^2)$$

we see that $u_*$ being the constrained minimizer is equivalent to requiring

$$\int 2\nabla u_* \cdot \nabla v + 2uv + \lambda q |u_*|^{q-2} u_* v \; dx = 0 \tag{1.4}$$

for every $v \in H^1_0$, with the additional requirement that $\|u_*\|_{L^q} = 1$. This equation can be interpreted as asking for $u_*$ to be a *weak solution* to the *nonlinear elliptic problem*

$$\begin{cases} -\Delta u_* + u_* + \dfrac{q\lambda}{2}|u_*|^{q-2} u_* = 0 \\ u_*|_{\partial\Omega} = 0 \end{cases} \tag{1.5}$$

Indeed, the techniques illustrated in these notes can be useful for solving similar nonlinear elliptic problems that admit variational formulations. Some of the historical highlights of this line of arguments include the solution of the Yamabe Problem[3], and the Sacks-Uhlenbeck Theorem[4].

**Remark 1.6.** An interesting consequence is the following: Let $U, V$ be bounded connected open domains with $C^1$ boundary, with $U \subsetneq V$ (and so that $V \setminus U$ contains an open set). Then the sharp constants $C_{*,U}$ and $C_{*,V}$ in the Sobolev embedding $H^1_0 \hookrightarrow L^q$ has the relationship $C_{*,U} < C_{*,V}$.

The non-strict inequality follows from the description

$$\frac{1}{C_*, U} = \inf\left\{\|u\|_{H^1} : u \in H^1_0(\Omega), \|u\|_{L^q} = 1\right\}.$$

Noting that if $u \in H^1_0(U)$, extending $u$ by zero across the boundary we get a function $\bar{u} \in H^1_0(V)$. So the infimum over $V$ must be smaller, and so we have $\frac{1}{C_{*,U}} \geq \frac{1}{C_{*,V}}$. This is essentially also what you proved on Problem 7 of Problem Set 5 of this course, for the eigenvalues.

To get the strict inequality: suppose that the two values were actually equal. This means that an optimizer for the $U$ problem is also an optimizer for the $V$ problem. Let $u_{*,U}$ be the optimizer, *which*

[3] See, T. Aubin, "Équations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire", J. Math. Pures Appl. (1976), or T. Aubin, *Some nonlinear problems in Riemannian geometry*, Springer (1998).

[4] J. Sacks and K. Uhlenbeck, "The Existence of Minimal Immersions of 2-Spheres", Annals of Math. (1981)

*we think of as an optimizer for the $V$ problem that vanishes on $V \setminus U$.* Looking at the nonlinear problem (1.5), we see that when $u_{*,U} \approx 0$, we can regard $u_{*,U}$ as satisfying an expression of the form $Lu_{*,U} = 0$ where

$$L = -\Delta + \underbrace{1 + \frac{q\lambda}{2}|u_*|^{q-2}}_{>0}.$$

The positivity of the "$c$" term means we can apply the strong maximum principle. But now we run into trouble in a neighborhood of $V \setminus U$ since the function vanishing on an open set implies it must vanish everywhere. This leads to a contradiction as $u_{*,U}$ has $L^q$ norm 1. So the inequality must be strict: that $u_{*,U}$ cannot actually be an optimizer for the problem in the larger domain.

In the proof of the theorem above, we showed that if $u_k \rightharpoonup u$ in $H^1_0$ then $u_k \to u$ in $L^q$. The argument given there can be abstracted as the following Lemma. We record it here since this argument will be used many times in the sequel.

**Lemma 1.7.** *Let $X, Y$ be reflexive Banach spaces, and $T : X \to Y$ a compact operator. Then $x_k \rightharpoonup x$ implies $Tx_k \to Tx$.*

*Proof.* Since $x_k$ converges weakly, the sequence must be norm bounded (uniform boundedness principle). Therefore by the compactness of $T$, the sequence $Tx_k$ has a convergent subsequence $Tx_{k_j} \to y \in Y$.

On the other hand, letting $\varphi \in Y^*$, the adjoint operator $T^* : Y^* \to X^*$ is continuous, and we have that $\varphi(Tx_k) = T^*\varphi(x_k) \to T^*\varphi(x) = \varphi(Tx)$ by weak convergence of $x_k \to x$, so $Tx$ is also a weak limit of $Tx_k$.

Since strong limits are weak limits, and weak limits are unique, this means that $Tx = y$.

Now, instead of applying this argument to $x_k$, we apply it to any subsequence of $x_k$, this shows that any subsequence of $Tx_k$ has a subsequence convergent to $Tx$. Standard metric space theory then shows this means $Tx_k \to Tx$. □

In the remainder of these notes, we will try to answer the existence problem for the optimizer of Sobolev embedding after relaxing the conditions. First we will relax the condition that $\Omega$ is bounded. Secondly (if time permits) we will relax the condition that $q < \frac{2d}{d-2}$ and study the limiting case $q = \frac{2d}{d-2}$. For historical reference, the codification of the concentration compactness argument for studying Sobolev embeddings was due to Gérard[5]

**Remark 1.8** (Notation for sequences)**.** In the following, to simplify notation, we will use $\vec{u}$ to denote a sequence, and $u_j$ to denote its individual terms. Given a strictly increasing function $\zeta : \mathbb{N} \to \mathbb{N}$, we will denote the subsequence traditionally labeled as $\{u_{\zeta_j}\}$ also by the notation $\vec{u} \circ \zeta$. The individual terms of this subsequence may be referred to as either $u_{\zeta(j)}$ or $u_{\zeta_j}$; the former can become easier to use when handling nested subsequences.

## §2: Non-compactness from translations

We will first tackle the question of the compactness of the Gagliardo-Nirenberg-Sobolev embedding $H^1(\mathbb{R}^d) \hookrightarrow L^q(\mathbb{R}^d)$. Here $d \geq 3$ and $q \in [2, \frac{2d}{d-2})$. Now, this embedding is known to be *non-compact*.

[5] P. Gérard, "Description du défaut de compacité de l'injection de Sobolev", ESAIM Control Optim. Calc. Var. 3 (1998).

**Example 2.1.** Let $u_0 \in C_c^\infty(B(0,1))$ be chosen with $\|u_0\|_{L^q} = 1$. Let $x_i = (2i, 0, 0, \ldots, 0) \in \mathbb{R}^d$. Set $u_i(x) = u_0(x - x_i)$. Then if $i \neq j$, the functions $u_i$ and $u_j$ have disjoint support, and so $\|u_i - u_j\|_{L^q} = \sqrt{2}$, therefore the sequence $\vec{u}$ has no Cauchy subsequences in $L^q$. On the other hand, by construction we have $\|u_i\|_{H^1} = \|u_0\|_{H^1}$ so the sequence is bounded in $H^1(\mathbb{R}^d)$. This shows that the Gagliardo-Nirenberg-Sobolev embedding cannot be compact.

Now, returning to the Proof of Theorem 1.1, we see that this poses the main difficulty in constructing an optimizer for the embedding problem. The Banach-Alaoglu step holds for general Hilbert spaces and doesn't care that our underlying domain is non-compact. And the uniqueness of the weak limit is also generally true. At issue is that we may not have a strong limit in $L^q$, due to the lack of compactness. Our discussion in this section will show that if we somehow "compensate" for the non-compactness arising from translations in $\mathbb{R}^d$, we can regain the existence of strong limits in $L^q$ and thereby obtain the existence of an optimizer.

§2.1 **Weak convergence modulo translations.—** Our goal is to compensate for the non-compact translations, so we start by defining a weak convergence concept that quotients out translations.

**Definition 2.2.** Given a function $u : \mathbb{R}^d \to \mathbb{R}$, we denote by $\tau_y u(x) := u(x - y)$ its translation by $y$.

Given a sequence $\vec{u}$ of functions on $\mathbb{R}^d$, and $\vec{y}$ a sequence of points in $\mathbb{R}^d$, by $\tau_{\vec{y}}\vec{u}$ we will understand the sequence of functions whose individual terms are $\tau_{y_k} u_k$.

**Definition 2.3** ($\tau$-weak convergence)**.** Given a sequence $\vec{u}$ in $H^1(\mathbb{R}^d)$, we say that the sequence converges *weakly modulo translations* to a function $u$, or that the sequence converges *$\tau$-weakly* to $u$, or in symbols $u_k \overset{\tau}{\rightharpoonup} u$, if for any $\varphi \in H^1$, we have

$$\lim_{k\to\infty} \sup_{y \in \mathbb{R}^d} \langle u_k - u, \tau_y \varphi \rangle = 0.$$

Here $\langle f, g \rangle = \int \nabla f \cdot \nabla g + f g$ is the $H^1$ inner product.

**Proposition 2.4.** *$u_k \overset{\tau}{\rightharpoonup} u$ if and only if for every sequence $\vec{y}$ in $\mathbb{R}^d$, the sequence whose terms are $\tau_{y_k}(u_k - u)$ converges to $0$ weakly.*

*Proof.* $\tau_{y_k}(u_k - u) \rightharpoonup 0 \iff \forall \varphi \langle \tau_{y_k}(u_k - u), \varphi \rangle \to 0 \iff \forall \varphi \langle u_k - u, \tau_{-y_k}\varphi \rangle = 0.$

For each $\varphi$, there exists a sequence $\vec{y}^\varphi$ such that

$$\left| \sup_{y \in \mathbb{R}^d} \langle u_k - u, \tau_y \varphi \rangle - \langle u_k - u, \tau_{y_k^\varphi} \varphi \rangle \right| < \frac{1}{k}.$$

Therefore $\forall \vec{y}\, \tau_{y_k}(u_k - u) \rightharpoonup 0 \implies u_k \overset{\tau}{\rightharpoonup} u$.

The reverse implication follows from observing that for any sequence $\vec{y}$, it holds from definition that $\langle u_k - u, \tau_{-y_k}\varphi \rangle \leq \sup_{y \in \mathbb{R}^d} \langle u_k - u, \tau_y \varphi \rangle$. □

**Corollary 2.5.** *If $u_k \overset{\tau}{\rightharpoonup} u$, then $u_k \rightharpoonup u$.*

**Proposition 2.6.** *Given $u_k \rightharpoonup 0$, if the sequence $\vec{y}$ does not converge to infinity, then there exists a subsequence indexed by $\zeta : \mathbb{N} \to \mathbb{N}$ such that $\tau_{y_{\zeta(j)}} u_{\zeta(j)} \rightharpoonup 0$.*

*Proof.* First, we note that weakly bounded sequences are bounded, and so $\vec{u}$ is a bounded sequence.

By Bolzano-Weierstrass, if $\vec{y}$ does not converge to infinity, then it contains a convergent subsequence $\vec{y} \circ \zeta \to y$. Since translations act continuously on $H^1$, we have that for every $\varphi \in H^1$, $\tau_{-y_{\zeta(j)}} \varphi \to \tau_{-y} \varphi$ strongly. Using that $\langle \tau_{y_{\zeta(j)}} u_{\zeta(j)}, \varphi \rangle = \langle u_{\zeta(j)}, \tau_{-y_{\zeta(j)}} \varphi \rangle$, we get

$$|\langle u_{\zeta(j)}, \tau_{-y_{\zeta(j)}} \varphi \rangle| \leq |\langle u_{\zeta(j)}, \tau_{-y} \varphi \rangle| + |\langle u_{\zeta(j)}, \tau_{-y_{\zeta(j)}} \varphi - \tau_{-y} \varphi \rangle|.$$

The first term converges to zero due to the assumed weak convergence $u_k \rightharpoonup 0$. The second term converges to zero since $\vec{u}$ is uniformly bounded and $\tau_{-y_{\zeta(j)}} \varphi \to \tau_{-y} \varphi$ strongly. □

The requirement that $\vec{y}$ does not converge to infinity is important.

**Example 2.7.** Let $u_k = \tau_{x_k} u_0$ be as in Example 2.1, then $u_k \rightharpoonup 0$. Let $y_k = -x_k$, then $\tau_{y_k} u_k = u_0$ is the constant sequence, and hence has no subsequence that converges weakly to zero.

This also shows that the converse of Corollary 2.5 is false.

On the other hand, we have the strong convergence implies $\tau$-weak convergence.

**Proposition 2.8.** *Given $u_k \to u$, then $u_k \overset{\tau}{\rightharpoonup} u$.*

*Proof.* We use that norm convergence is translation invariant, and so $u_k \to u \implies \tau_{y_k}(u_k - u) \to 0$ in norm, and hence weakly, for any sequence $\vec{y}$. By Proposition 2.4 this proves the proposition. □

**Example 2.9.** The converse of the above proposition is false. Let $u_k$ be as in Example 2.1; recall that they have pairwise disjoint support. Define

$$v_k = \frac{1}{\sqrt{k}} \sum_{j=1}^{k} u_j.$$

Due to the pairwise disjoint support, we have that $\|v_k\|_{H^1} = \|u_0\|_{H^1}$, and so clearly $v_k \not\to 0$.

I claim however that $v_k \overset{\tau}{\rightharpoonup} 0$. Let $\varphi \in H^1$, and $\epsilon > 0$. There exists a ball $B$ of radius $R$ sufficiently large such that $\|\varphi\|_{H^1(B^c)} \leq \epsilon$. By construction, any translations of this ball $B$ can fit at most $R/2$ of the $u_k$ in it simultaneously. And so we have

$$\langle v_k, \tau_y \varphi \rangle = \langle v_k, (\tau_y \varphi)_{\tau_y B} \rangle + \langle v_k, (\tau_y \varphi)|_{\tau_y B^c} \rangle.$$

The first term is bounded by

$$|\langle v_k, (\tau_y \varphi)_{\tau_y B} \rangle| \leq \frac{R}{2\sqrt{k}} \|u_0\|_{H^1} \|\varphi\|_{H^1}.$$

The second term is bounded by

$$|\langle v_k, (\tau_y \varphi)_{\tau_y B^c} \rangle| \leq \|u_0\|_{H^1} \epsilon.$$

So first choosing $\epsilon$ sufficiently small, and then choosing $k$ sufficiently large to compensate for $R$, we see that $|\langle v_k, \tau_y \varphi\rangle|$ can be made arbitrarily small (for sufficiently large $k$), independently of $y$. This shows exactly that $v_k \overset{\tau}{\rightharpoonup} 0$.

**Example 2.10.** Observe however, that the functions $v_k$ converges strongly to 0 in $L^q$, for $q > 2$. Indeed, using the disjoint support condition, we have

$$\|v_k\|_{L^q}^q = \sum_{j=1}^{k} \frac{1}{k^{q/2}} \|u_j\|_{L^q}^q = \frac{1}{k^{(q-2)/2}} \|u_0\|_{L^q}^q \searrow 0.$$

The previous example provides some evidence for the following theorem, which is the main result of this section. It indicates that the notion of $\tau$-weak convergence in $H^1$ is intimately tied to strong convergence in $L^q$.

**Theorem 2.11.** *Let $\vec{u}$ be a bounded sequence of $H^1(\mathbb{R}^d)$ functions. Then the following two statements are equivalent:*

1. $u_k \overset{\tau}{\rightharpoonup} 0$;
2. $\|u_k\|_{L^q} \to 0$ *for some* $q \in (2, \frac{2d}{d-2})$.

*Proof that 1 $\Longrightarrow$ 2.* This proof is using essentially the same argument that appeared in the previous example.

We will divide up $\mathbb{R}^d$ into a countable collection of disjoint unit cubes $Q_\mu$, indexed by $\mu \in \mathbb{N}$. Given a function $f : \mathbb{R}^d \to \mathbb{R}$, we will denote by $\pi_\mu f$ its restriction to $Q_\mu$. We observe the following, due to the disjoint nature of the cubes:

- $\|f\|_{H^1(\mathbb{R}^d)} = \left(\sum_\mu \|\pi_\mu f\|_{H^1(Q_\mu)}^2\right)^{1/2}$;
- $\|f\|_{L^q(\mathbb{R}^d)} = \left(\sum_\mu \|\pi_\mu f\|_{L^q(Q_\mu)}^q\right)^{1/q}$.

Additionally, we have that on each of the unit cubes, we can apply Sobolev embedding: there exists some constant $C$ such that $\|\pi_\mu f\|_{L^q} \le C\|\pi_\mu f\|_{H^1}$.

Now consider the function $u_k$. We have, since $2 < q < \infty$, we can apply Hölder's inequality to obtain the interpolation

$$\begin{aligned}
\|u_k\|_{L^q(\mathbb{R}^d)} &= \left(\sum_\mu \|\pi_\mu u_k\|_{L^q(Q_\mu)}^q\right)^{1/q} \\
&\le \left(\sum_\mu \|\pi_\mu u_k\|_{L^q(Q_\mu)}^2\right)^{1/q} \left(\sup_\mu \|\pi_\mu u_k\|_{L^q(Q_\mu)}\right)^{1-2/q}
\end{aligned}$$

and now by Sobolev embedding on the terms inside the first brackets

$$\leq C^{2/q}\left(\sum_{\mu}\|\pi_\mu u_k\|^2_{H^1(Q_\mu)}\right)^{1/q}\left(\sup_\mu\|\pi_\mu u_k\|_{L^q(Q_\mu)}\right)^{1-2/q}$$
$$\leq C^{2/q}\|u_k\|^{2/q}_{H^1(\mathbb{R}^d)}\left(\sup_\mu\|\pi_\mu u_k\|_{L^q(Q_\mu)}\right)^{1-2/q}.$$

By hypothesis, we have that $\|u_k\|_{H^1}$ is uniformly bounded. Therefore to prove our desired conclusion, it is enough to show that

$$\sup_\mu\|\pi_\mu u_k\|_{L^q(Q_\mu)} \to 0. \tag{2.12}$$

Now, for each $u_k$, there exists a cube $Q_{\mu_k}$ such that

$$\|\pi_{\mu_k} u_k\|_{L^q(Q_{\mu_k})} \geq \frac{1}{2}\sup_\mu\|\pi_\mu u_k\|_{L^q(Q_\mu)}$$

by the definition of the supremum. Let $y_k$ be such that $\tau_{y_k}$ brings the cube $Q_{\mu_k}$ to the location of $Q_1$, so that

$$\pi_1\tau_{y_k}u_k = \tau_{y_k}\pi_{\mu_k}u_k.$$

Now consider the sequence $\tau_{y_k}u_k$: by our hypothesis $u_k \overset{\tau}{\rightharpoonup} 0$ and hence $\tau_{y_k}u_k \rightharpoonup 0$, and hence $\pi_1\tau_{y_k}u_k \rightharpoonup 0$. On the other hand, since $Q_1$ is a bounded domain, we know that the embedding $H^1(Q_1) \hookrightarrow L^q(Q_1)$ is compact, and hence by Lemma 1.7 we have that $\|\pi_1\tau_{y_k}u_k\|_{L^q} \to 0$. By construction, as $\|\pi_1\tau_{y_k}u_k\|_{L^q} = \|\pi_{\mu_k}u_k\|_{L^q} \geq \frac{1}{2}\sup_\mu\|\pi_\mu u_k\|_{L^q}$, this shows (2.12) holds as desired. □

*Proof that 2 $\Longrightarrow$ 1.* Let $\{y_k\}$ be an arbitrary sequence in $\mathbb{R}^d$. Let $\{v_j\}$ be an arbitrary subsequence of $\{\tau_{y_k}u_k\}$. By Banach-Alaoglu, $\{v_j\}$ has a weakly convergent subsequence in $H^1$; let $v$ denote said weak limit. Let $A$ be some round ball. Applying Rellich-Kondrachav and Lemma 1.7 (using that $v_j \to 0$ strongly in $L^q$) we see that $v|_A = 0$.

Now given $\varphi \in H^1_0(A)$, extended by 0 to the exterior of $A$, consider the sequence of numbers $\langle\tau_{\vec{y}}\vec{u},\varphi\rangle$. The argument given in the previous paragraph shows that any subsequence of this sequence has a further subsequence that converges to 0. This shows that this sequence converges to 0. Now using that $H^1$ functions of compact support is dense in $H^1$, we see this shows that $\tau_{\vec{y}}\vec{u}$ converges weakly to zero. Since $\vec{y}$ is arbitrary, by Proposition 2.4 this shows that $u_k \overset{\tau}{\rightharpoonup} 0$. □

**Corollary 2.13.** *If a sequence of $H^1(\mathbb{R}^d)$ functions $u_k \overset{\tau}{\rightharpoonup} u$, then $u_k$ converges to $u$ strongly in $L^q(\mathbb{R}^d)$ for any $q \in (2, \frac{2d}{d-2})$.*

**§2.2 Cores and profiles.—** In view of Corollary 2.13, our next task is to clarify under what situations can we obtain $\tau$-weak convergence of a derived sequence, starting from a bounded sequence of $H^1(\mathbb{R}^d)$ functions. The following example shows that the situation can be complicated.

**Example 2.14.** Fix $u_0 \in C_0^\infty(B(0,1))$. Consider a sequence of points $x_k \in \mathbb{R}^d$ diverging to infinity. Now define a sequence of functions

$$u_k = u_0 + \tau_{x_k} u_0.$$

I claim that this bounded sequence has no $\tau$-weak limit.

Indeed, were $u_*$ the $\tau$-weak limit of $u_k$, then $u_k \rightharpoonup u_*$ necessarily, which would imply that $u_* = u_0$ (using that $x_k$ diverges to infinity and so $\tau_{x_k} u_0 \rightharpoonup 0$). However, $\tau_{-x_k}(u_k - u_0) = u_0 \not\rightharpoonup 0$. This show that bounded sequences need not have $\tau$-weak limits. In fact, our construction shows something stronger: not only does $\{u_k\}$ have no $\tau$-weak limits, there are no *subsequences* of $\{u_k\}$ with $\tau$-weak limits.

In the preceding example, the lack of $\tau$-weak convergence is essentially due to a "portion" of the sequence $u_k$ "moving away to infinity in a coherent way". One can generalize this situation to having countably many parts that diverge to infinity at different rates. This motivates us to our main construction.

**Definition 2.15** ($\tau$-core)**.** By a *$\tau$-core* we refer to a sequence $\vec{\phi}$ of the form $\phi_k = \tau_{y_k}\phi$, where $\phi \not\equiv 0$. Here the fixed function $\phi$ is called the *shape* of the core, and the sequence $\vec{y}$ its *progression*.

**Definition 2.16** ($\tau$-equivalence of cores)**.** Given two $\tau$-cores $\vec{\phi}$ and $\vec{\psi}$ with shapes $\phi$ and $\psi$ and progressions $\vec{y}$ and $\vec{z}$ respectively, we say they are *$\tau$-equivalent*, denoted $\vec{\phi} \overset{\tau}{\approx} \vec{\psi}$, if $\vec{y} - \vec{z}$ is convergent with limit $w$, and that $\tau_w \phi = \psi$.

Below we will use the usual notation for equivalence classes: $[\vec{\phi}] := \{\vec{\psi} : \vec{\psi} \overset{\tau}{\approx} \vec{\phi}\}$.

**Remark 2.17.** As a direct consequence of the definition, whenever $\vec{\phi} \overset{\tau}{\approx} \vec{\psi}$, we also find $\vec{\phi} - \vec{\psi} \to 0$ strongly in $H^1$. Additionally, note that if $\vec{\phi} \overset{\tau}{\approx} \vec{\psi}$ and letting $\phi$ and $\psi$ be their respective shapes, necessarily $\|\phi\| = \|\psi\|$ for any $\tau$-invariant norm on functions. This means that it is meaningful to write $\|[\vec{\phi}]\|$ for the norm of an equivalence class of $\tau$-cores.

We have required above that $\tau$-cores be necessarily a non-trivial sequence. This is because that in $H^1$, the only function satisfying $\tau_w f = f$ with $w \neq 0$ is the zero function. And so given a $\tau$-core $\vec{\phi}$ and a choice of shape $\phi$, the corresponding progression $\vec{y}$ is uniquely determined. (One can of course translate each $y_k$ by a fixed vector $-w$, and select $\tau_w \phi$ as the shape; this defines the same $\tau$-core just with a different parametrization.)

An important property of equivalence is the following:

**Lemma 2.18.** *Let $\vec{\phi}$ and $\vec{\psi}$ be equivalent $\tau$-cores, with shapes $\phi, \psi$ and progressions $\vec{y}, \vec{z}$ respectively. Suppose a bounded sequence $\vec{u}$ of $H^1$ functions is such that $\tau_{-\vec{y}}\vec{u} \rightharpoonup \phi$, then $\tau_{-\vec{z}}\vec{u} \rightharpoonup \psi$.*

*Proof.* Let $v \in H^1$. Denote $\vec{w} = \vec{y} - \vec{z}$; $\tau$-equivalence implies that $w_k \to w$. Our hypothesis states that

$$\begin{aligned} \langle \tau_{-y_k} u_k - \phi, v\rangle \to 0 \implies \langle \tau_{-w_k}\tau_{-z_k} u_k - \phi, v\rangle \to 0 \implies \\ \langle \tau_{-z_k} u_k - \tau_{w_k}\phi, \tau_{w_k} v\rangle \to 0 \implies \langle \tau_{-z_k} u_k - \psi, \tau_{w_k} v\rangle + \langle \psi - \tau_{w_k}\phi, \tau_{w_k} v\rangle \to 0 \end{aligned}$$

The second term converges to zero since $\tau_{w_k}\phi \to \psi$ strongly by assumption. Similarly, since $\tau_{w_k} v \to \tau_w v$ strongly, we conclude then

$$\langle \tau_{-z_k} u_k - \psi, \tau_w v\rangle \to 0$$

and since $v$ is arbitrary the conclusion follows. □

Next we define the opposite concept to equivalence.

**Definition 2.19** ($\tau$-orthogonality of cores)**.** Two $\tau$-cores $\vec{\phi}$ and $\vec{\psi}$ are said to be $\tau$-*orthogonal*, denoted $\vec{\phi} \overset{\tau}{\perp} \vec{\psi}$, if their respective progressions $\vec{y}$ and $\vec{z}$ satisfies $\lim |y_k - z_k| = +\infty$.

The following lemma is immediate.

**Lemma 2.20.** *Given $\vec{\phi} \overset{\tau}{\perp} \vec{\psi}$, and $\vec{\phi}' \overset{\tau}{\approx} \vec{\phi}$ and $\vec{\psi}' \overset{\tau}{\approx} \vec{\psi}$, then $\vec{\phi}' \overset{\tau}{\perp} \vec{\psi}'$ also. In other words, $\tau$-orthogonality is a well-defined notion for* equivalence classes *of $\tau$-cores.*

As seen in Example 2.14, we want to use this notion of cores to capture the situation where the sequence $\vec{u}$ is made up of several coherent parts that are flying apart from each other. This motivates the following definition.

**Definition 2.21** ($\tau$-profile)**.** A $\tau$-*profile* $\mathcal{P}$ is a set of pairwise $\tau$-orthogonal equivalence classes of $\tau$-cores.

**Definition 2.22** (Subordinance)**.** We say that a $\tau$-profile $\mathcal{P}$ is *subordinate to* a bounded sequence $\vec{u}$ of $H^1$ functions if, for every[6] $[\vec{\phi}] \in \mathcal{P}$ with progression $y_k$ and shape $\phi$, we have $\tau_{-y_k} u_k \rightharpoonup \phi$.

Denoting by $\mathfrak{P}[\vec{u}]$ the set of all $\tau$-profiles subordinate to $\vec{u}$, we can partial order the elements by inclusion. An element $\mathcal{P} \in \mathfrak{P}[\vec{u}]$ is said to be *maximal* if it is an maximal element with respect to this partial order.

In general, $\tau$-profiles can contain many elements. For example, let $y_0$ be a fixed non-vanishing vector. Observe that if $\lambda \neq \nu \in \mathbb{R}$, then setting $z_k = k\lambda y_0$ and $z_k' = k\nu y_0$, we find that $\lim |z_k - z_k'| = +\infty$. So for any fixed $\phi$, the $\tau$-cores $\phi_k = \tau_{z_k}\phi$ and $\phi_k' = \tau_{z_k'}\phi$ are $\tau$-orthogonal. This allows us to build a profile with uncountably many elements.

It turns out, however, that if the profile is *subordinate* to a bounded sequence $\vec{u}$ of $H^1$ functions, then necessarily the profile has at most countably many elements, and that it enjoys a certain degree of boundedness.

**Lemma 2.23.** *Given a bounded sequence $\{u_k\}$ of $H^1$ functions and $\mathcal{P}$ a subordinate $\tau$-profile, then $\mathcal{P}$ is countable and*[7]

$$\sum_{[\vec{\phi}] \in \mathcal{P}} \|[\vec{\phi}]\|_{H^1}^2 \leq \limsup \|u_k\|_{H^1}^2.$$

*Proof.* We use the following fact that is deeply tied to the Hilbert space structure of $H^1$: for any sequence of functions $f_k \rightharpoonup f$,

$$\limsup \|f_k - f\|^2 = \limsup(\|f_k\|^2 - 2\langle f, f_k\rangle + \|f\|^2) = \limsup \|f_k\|^2 - \|f\|^2. \tag{2.24}$$

[6]This definition holds regardless of choice of representative from the equivalence class, due to Lemma 2.18.

[7]Recall from Remark 2.17 that the norm of a $\tau$-core is well-defined.

(The lim inf version appeared earlier in these notes too!) This says that in the limit, subtracting the weak limit reduces the norm. Now let $\mathcal{P}_f \subset \mathcal{P}$ be any finite subset. Consider

$$v_k := u_k - \sum_{[\vec{\phi}]\in\mathcal{P}_f} \tau_{y_k}\phi;$$

here $\phi$ is the shape of $\vec{\phi}$ and $\vec{y}$ its progression. Since $\mathcal{P}_f$ is finite, the sum is well-defined up to a choice of representative from each equivalence class. If $[\vec{\psi}] \in \mathcal{P}\setminus\mathcal{P}_f$, then by definition $\vec{\psi} \overset{\tau}{\perp} \vec{\phi}$ for every $[\vec{\phi}] \in \mathcal{P}_f$. Therefore, if $\vec{\psi}$ has shape $\psi$ and progression $\vec{w}$, we have

$$\tau_{-w_k}\left(\sum_{[\vec{\phi}]\in\mathcal{P}_f} \tau_{y_k}\phi\right) = \sum_{[\vec{\phi}]\in\mathcal{P}_f} \tau_{y_k - w_k}\phi \rightharpoonup 0$$

as the sum is finite, and hence $\tau_{-w_k} v_k \rightharpoonup \psi$ also. Thus we have, by (2.24),

$$0 \le \limsup \|v_k - \tau_{w_k}\psi\|^2 = \limsup \|v_k\|^2 - \|\psi\|^2.$$

Note that the conclusion $0 \le \limsup \|v_k\|^2 - \|\psi\|^2$ is reached regardless of the choice of representatives from each equivalence class. By induction, this shows that for any finite subset $\mathcal{P}_f$

$$\limsup \|u_k\|^2 \ge \sum_{[\vec{\phi}]\in\mathcal{P}_f} \|[\vec{\phi}]\|^2.$$

This uniform boundedness over finite sums implies then

1. $\mathcal{P}$ has at most countably many equivalence classes; and
2. the possibly infinite sum over the entire $\mathcal{P}$ is well-defined and also bounded by $\limsup \|u_k\|^2$.

□

The Lemma 2.23 provides a powerful control on the sizes of the $\tau$-profile subordinate to a sequence $\vec{u}$; however, it does not guarantee that a formal sum of the form

$$\sum_{[\vec{\phi}]\in\mathcal{P}} \phi_k \tag{2.25}$$

converges. (Here we are considering $k$ as fixed, and that the sum is over all equivalence classes in $\mathcal{P}$, with a representative arbitrarily selected for each class.) This is because that formally, the Hilbert space norm of the above sum would look like

$$\left\| \sum_{[\vec{\phi}]\in\mathcal{P}} \tau_{y_j}\phi_k \right\|^2 = \sum_{[\vec{\phi}]\in\mathcal{P}} \left\|[\vec{\phi}]\right\|^2 + \sum_{\substack{[\vec{\psi}],[\vec{\phi}]\in\mathcal{P} \\ [\vec{\psi}]\neq[\vec{\phi}]}} \langle \psi_k, \phi_k\rangle.$$

The first term, the diagonal sums, are controlled by Lemma 2.23. The second term, the cross sums, may never-the-less be unbounded. At issue is the fundamental fact that on the infinite sequence space, the $\ell^1$ norm is strictly stronger than the $\ell^2$ norm, and so the bounds provided by Lemma 2.23 (as opposed to one is given through an $\ell^1$ sum) is not enough for norm-convergence.

However, we see in some sense that "in the limit" as $k \nearrow \infty$ the cross sums should drop out: this is because the progressions for $\vec{\psi}$ and $\vec{\phi}$ are supposed to diverge with their distance tending to infinity, whenever we are looking at two representatives from distinct equivalence classes, due to the $\tau$-orthogonality assumption. So our expectations is that at least eventually (in the sense that $k \to \infty$) a formal sum of the form (2.25) should be meaningful.

The following Selection Theorem makes precise this intuition. In fact we prove something stronger: that there is a way to choose representatives in such a way that, *for that specific choice of representatives*, the sum (2.25) converges in $H^1$ for every $k$. Furthermore, this convergence is uniform in $k$.

**Theorem 2.26** (Selection Theorem)**.** *Given a bounded sequence $\vec{u}$ in $H^1$ and a $\tau$-profile $\mathcal{P}$ subordinate to it. Then there exists a choice of representatives $[\vec{\phi}] \to \vec{\phi}$ for each of the equivalence classes in $\mathcal{P}$, such that*

$$k \mapsto \sum_{[\vec{\phi}]\in\mathcal{P}} \phi_k$$

*converges in $H^1$ and* uniformly in $k$*; the latter means that for every $\epsilon > 0$ there exists a co-finite subset $\mathcal{P}' \subseteq \mathcal{P}$ such that, with the specified choice of representatives,*

$$\sup_k \Big\| \sum_{[\vec{\phi}]\in\mathcal{P}'} \phi_k \Big\|_{H^1} < \epsilon.$$

**Remark 2.27.** Note that the theorem is automatically true if $\mathcal{P}$ is finite, due to the comparability of all $\ell^p$ norms in finite dimensions. At issue is only the situation when there are infinitely many equivalence classes inside $\mathcal{P}$.

*Proof of Theorem 2.26.* Our strategy is to start with an arbitrary choice of representatives for each equivalence class. We then, in order, adjust each representative in such a way that the *tail* of their corresponding progression remains unaffected; in this way the adjusted $\tau$-core remains clearly in the same equivalence class. We claim that an iterative adjustment of this sort can guarantee the required summability.

For convenience, assume an initial choice of representatives have been made. Since $\mathcal{P}$ maybe assumed to be countably infinite (the finite case is trivial), we enumerate the corresponding shapes as $\phi_\alpha$ with $\alpha \in \mathbb{N}$, and the corresponding progressions as $y_k^\alpha$. Note that since the profiles correspond to mutually $\tau$-orthogonal equivalence classes, we must have that whenever $\alpha \neq \beta$ that $|y_k^\alpha - y_k^\beta| \to +\infty$. In particular, this shows that fixing $\alpha \neq \beta$, we have that

$$\langle \tau_{y_k^\alpha} \phi_\alpha, \tau_{y_k^\beta} \phi_\beta \rangle \to 0. \tag{2.28}$$

This equation allows us to make the following definition of a function $J : \mathbb{N} \to \mathbb{N}$.

$$J(\gamma) = \min\left\{ j \in \mathbb{N} : \sum_{\substack{\alpha\neq\beta \\ \alpha,\beta\leq\gamma}} |\langle \tau_{y_k^\alpha} \phi_\alpha, \tau_{y_k^\beta} \phi_\beta \rangle| \leq \frac{1}{\gamma} \quad \forall k \geq j \right\}. \tag{2.29}$$

Notice by definition that $J(\gamma)$ is increasing, and finite for each $\gamma$.

For each $\alpha$, we will define a new progression $\{z_k^\alpha\}$ using the following iterative procedure: we will assume that $\{z_k^\beta\}$ has been defined for all $\beta < \alpha$.

1. For $k \geq J(\alpha)$, we will set $z_k^\alpha = y_k^\alpha$. This ensures $\tau_{z_k^\alpha}\phi_\alpha \overset{\tau}{\approx} \tau_{y_k^\alpha}\phi_\alpha$.
2. For $k < J(\alpha)$, select $z_k^\alpha$ sufficiently far from $z_k^\beta$, for every $\beta < \alpha$, to ensure that

$$\sum_{\beta<\alpha} |\langle \tau_{z_k^\beta}\phi_\beta, \tau_{z_k^\alpha}\phi_\alpha\rangle| \leq \frac{1}{2\alpha}2^{-\alpha}. \tag{2.30}$$

It remains to verify that this choice ensures uniform convergence. We will show that for every $\epsilon > 0$, there exists some $\gamma$ such that

$$\Big\| \sum_{\alpha>\gamma} \tau_{z_k^\alpha}\phi_\alpha \Big\|_{H^1}^2 < \epsilon$$

holds for every $k$. Since we are working in a Hilbert space, we can write the left hand side as

$$\sum_{\alpha,\beta>\gamma} \langle \tau_{z_k^\alpha}\phi_\alpha, \tau_{z_k^\beta}\phi_\beta\rangle = \sum_{\alpha>\gamma} \|\tau_{z_k^\alpha}\phi_\alpha\|_{H^1}^2 + \sum_{\substack{\alpha,\beta>\gamma\\ \alpha\neq\beta}} \langle \tau_{z_k^\alpha}\phi_\alpha, \tau_{z_k^\beta}\phi_\beta\rangle.$$

We treat the diagonal and off-diagonal sums separately. For the diagonal sum, using Lemma 2.23, we see that choosing $\gamma$ sufficiently large can ensure that it is $< \frac{1}{2}\epsilon$.

For the off-diagonal sums, we will split it into two parts. Recall that for now we are considering $k$ to be fixed but arbitrary. For each $\alpha, \beta$ that occur in the sum, we can ask whether $J(\alpha)$ and $J(\beta)$ is larger than $k$ or not. We treat separately the case where *both* $J(\alpha) \leq k$ and $J(\beta) \leq k$, and the case where at least one is $> k$.

Where at least one is $> k$, we can without loss of generality assume $\alpha > \beta$. And since $J$ is increasing, this guarantees that $J(\alpha) > k$. Then the corresponding sum is (doubled to account for the case $\beta > \alpha$)

$$\begin{aligned} 2\Bigg|\sum_{\substack{\alpha>\beta>\gamma\\ J(\alpha)>k}} \langle \tau_{z_k^\alpha}\phi_\alpha, \tau_{z_k^\beta}\phi_\beta\rangle\Bigg| &\leq 2\sum_{\substack{\alpha>\beta>\gamma\\ J(\alpha)>k}} \Big|\langle \tau_{z_k^\alpha}\phi_\alpha, \tau_{z_k^\beta}\phi_\beta\rangle\Big| \\ &\leq 2\sum_{\substack{\alpha:J(\alpha)>k\\ \alpha>\gamma}} \left(\sum_{\beta:\beta<\alpha} \Big|\langle \tau_{z_k^\alpha}\phi_\alpha, \tau_{z_k^\beta}\phi_\beta\rangle\Big|\right) \end{aligned}$$

the inner sum we can control using our earlier construction: by (2.30) we find

$$\leq 2\sum_{\substack{\alpha>\gamma\\ J(\alpha)>k}} \frac{1}{2\alpha}2^{-\alpha} \leq \frac{1}{\gamma}2^{-\gamma}.$$

When both $J(\alpha)$ and $J(\beta)$ are $\leq k$, we will apply the definition (2.29) for the function $J$. For each $k$, define $\kappa_0 = \sup\{\kappa : J(\kappa) \leq k\}$. (Note that $\kappa_0$ may be $\infty$; we will pretend it is finite for now, and hence $\kappa_0$

is in fact defined by a max and $J(\kappa_0) \leq k$ also; the infinity case requires some minor modifications left to the reader.) As $k \geq J(\kappa_0)$, this means, since $J$ is increasing, that $\alpha, \beta \leq \kappa_0$ implies $J(\alpha), J(\beta) \leq k$. Since $\kappa_0$ is defined by a supremum, the reverse implication also hold. Hence the sum

$$\sum_{\substack{\alpha,\beta>\gamma \\ \alpha\neq\beta \\ J(\alpha),J(\beta)\leq k}} (\text{ formula }) = \sum_{\substack{\gamma<\alpha,\beta\leq\kappa_0 \\ \alpha\neq\beta}} (\text{ formula }).$$

So applying (2.29) and noting that when $k \geq J(\alpha), J(\beta)$ we have $z_k^\alpha = y_k^\alpha$ and $z_k^\beta = y_k^\beta$ by construction, we conclude

$$\left| \sum_{\substack{\alpha,\beta>\gamma \\ \alpha\neq\beta \\ J(\alpha),J(\beta)\leq k}} \langle \tau_{z_k^\alpha} \phi_\alpha, \tau_{z_k^\beta} \phi_\beta \rangle \right| \leq \frac{1}{\kappa_0} \leq \frac{1}{\gamma}.$$

Putting the two parts together, we have that the off-diagonal sums are bounded by $\frac{1}{\gamma} + \frac{1}{\gamma} 2^{-\gamma} \leq \frac{2}{\gamma}$. So choosing $\gamma$ sufficiently large such that $\frac{2}{\gamma} < \frac{1}{2}\epsilon$, we can combine with the estimates on the diagonal term to prove uniform convergence. □

**Remark 2.31** (Notation)**.** In view of the Selection Theorem, whenever $\mathcal{P}$ is a $\tau$-profile subordinate to some bounded sequence $\vec{u}$ of $H^1$ functions, we will write $\vec{\sum}\mathcal{P}$ for the sequence of $H^1$ functions given by the formal sum (2.25), which we can assume to converge due to a good choice of representatives.

**Remark 2.32.** We have actually proven more than we claimed. The argument above is based on providing precise and decaying control on the size of the non-diagonal terms, and hence it in fact shows that

$$\left\|(\Sigma\mathcal{P})_k\right\|_{H^1}^2 \to \sum_{[\vec{\phi}]\in\mathcal{P}} \|[\vec{\phi}]\|_{H^1}^2.$$

We conclude this section with a discussion of maximal $\tau$-profiles.

**Lemma 2.33.** *$\mathcal{P}$ is a maximal $\tau$-profile subordinate to the sequence $\vec{\sum}\mathcal{P}$.*

*Proof.* First, let $[\vec{\phi}] \in \mathcal{P}$, with shape $\phi$ and progression $\vec{y}$. We need to show that $\tau_{-\vec{y}}\vec{\sum}\mathcal{P} \rightharpoonup \phi$. Using the uniform convergence of the sum $\vec{\sum}\mathcal{P}$, we can find a finite subset $\mathcal{P}' \subset \mathcal{P}$ containing $[\vec{\phi}]$ such that $\|\vec{\sum}\mathcal{P}' - \vec{\sum}\mathcal{P}\|_{H^1}$ is arbitrarily and uniformly small. The finite sum $\tau_{-\vec{y}}\vec{\sum}\mathcal{P}'$ can be written as $\phi + \tau_{-\vec{y}}\vec{\sum}(\mathcal{P}' \setminus \{[\vec{\phi}]\})$. The due to the pair-wise $\tau$-orthogonality of the classes in $\mathcal{P}$, the latter sum can be expanded into a finite sum of terms weakly convergent to 0. A standard argument shows then $\tau_{-\vec{y}}\vec{\sum}\mathcal{P} \rightharpoonup \phi$ as desired, this shows that $\mathcal{P}$ is subordinate to $\vec{\sum}\mathcal{P}$.

Next we need to show that $\mathcal{P}$ is maximal. The argument is largely the same as the previous paragraph. Fix $\vec{y}$ such that it would be orthogonal to the progressions of all classes in $\mathcal{P}$. It suffices to show that $\tau_{-\vec{y}}\vec{\sum}\mathcal{P} \rightharpoonup 0$. As above, we first note that $\vec{\sum}\mathcal{P}$ can be arbitrarily well-approximated by a finite sum $\vec{\sum}\mathcal{P}'$. For the finite sum we note that the orthogonality of $\vec{y}$ implies $\tau_{-\vec{y}}\vec{\sum}\mathcal{P}' \rightharpoonup 0$, as a finite sum of terms each of which are weakly convergent to zero. Together a standard argument shows the desired conclusion. □

**Corollary 2.34.** *Given a bounded sequence $\vec{u}$ in $H^1$, and a subordinate $\tau$-profile $\mathcal{P}$. Then $\mathcal{P}$ is non-maximal if and only if $\vec{u} - \vec{\sum}\mathcal{P}$ admits a non-empty subordinate $\tau$-profile.*

*Proof.* The forward implication follows from the previous Lemma: if $\mathcal{P}$ is non-maximal, there exists a $\tau$-core orthogonal to $\mathcal{P}$, with shape $\phi$ and progression $\vec{y}$, such that $\tau_{-\vec{y}}\vec{u} \rightharpoonup \phi$. But by the previous lemma we have $\tau_{-\vec{y}}\vec{\sum}\mathcal{P} = 0$. This shows that $\{[\vec{\phi}]\} \in \mathfrak{P}[\vec{u} - \vec{\sum}\mathcal{P}]$.

For the reverse implication, we need to show that if $\{[\vec{\phi}]\} \in \mathfrak{P}[\vec{u} - \vec{\sum}\mathcal{P}]$, then $[\vec{\phi}] \overset{\tau}{\perp} \mathcal{P}$. Suppose not, then there exists $[\vec{\psi}] \in \mathcal{P}$ such that, writing $\vec{y}$ and $\vec{z}$ the progressions of $\vec{\phi}$ and $\vec{\psi}$ respectively, that $\vec{y} - \vec{z}$ has a bounded subsequence indexed by $\zeta$. Using Heine-Borel, by taking a further subsequence we may assume $(\vec{y} - \vec{z}) \circ \zeta$ converges to $w$. By hypothesis, $\tau_{-\vec{z}}(\vec{u} - \vec{\sum}\mathcal{P}) \rightharpoonup 0$. Then so does the subsequence indexed by $\zeta$. The strong convergence of $\vec{y} - \vec{z}$ implies (similarly to the proof of Lemma 2.18) that $\left(\tau_{-\vec{y}}\left(\vec{u} - \vec{\sum}\mathcal{P}\right)\right) \circ \zeta \rightharpoonup 0$ also. On the other hand, that $[\vec{\phi}]$ is a $\tau$-core of a $\tau$-profile subordinate to $\vec{u} - \vec{\sum}\mathcal{P}$ implies that $\tau_{-\vec{y}}(\vec{u} - \vec{\sum}\mathcal{P}) \rightharpoonup \phi \neq 0$. This gives a contradiction. And so $\{[\vec{\phi}]\}$ is $\tau$-orthogonal to $\mathcal{P}$, which implies by the previous lemma that $\tau_{-\vec{y}}\vec{\sum}\mathcal{P} \rightharpoonup 0$, and hence $\{[\vec{\phi}]\} \in \mathfrak{P}[\vec{u}]$ and so $\mathcal{P}$ is non-maximal. □

**§2.3 Concentration Compactness.—** Intuitively, in view of the Selection Theorem, what we would like is to start with a bounded sequence $\vec{u}$ of $H^1$ functions, find a *maximal* $\tau$-profile subordinate to $\vec{u}$, and hope that $\vec{u} - \vec{\sum}\mathcal{P}$ is a sequence that $\tau$-weakly converges to 0. This, however, is too naïve.

**Example 2.35.** Fix $\phi \in H^1$. Let $u_{2n} = 0$ and $u_{2n+1} = \phi$. Then $\vec{u}$ is bounded. Let $\vec{y}$ be an arbitrary sequence. If a subsequence $\vec{y} \circ \zeta$ diverges to infinity, then the subsequence $(\tau_{-\vec{y}}\vec{u}) \circ \zeta \rightharpoonup 0$. So if $\tau_{-\vec{y}}\vec{u}$ were to have a non-trivial weak limit, it must remain bounded. Let $w$ be an accumulation point of the "odd" subsequence of $\vec{y}$. Observe then

$$\langle \tau_{-\vec{y}}\vec{u}, \tau_{-w}\phi \rangle$$

is both frequently zero and frequently near the value $\|\phi\|^2$. And so we've actually shown that there can be *no* $\tau$-profiles subordinate to $\vec{u}$.

On the other hand, one also sees that $\|u_k\|_{L^p} \not\rightarrow 0$, and so by Theorem 2.11 we see that $u_k$ does not $\tau$-weakly converge to 0.

In a sense, this is not entirely unexpected: recall that Banach-Alaoglu Theorem only guarantees that a bounded sequence has a weakly convergent *sub*sequence. And indeed, in the example above, the "even" subsequence is identically zero, and the "odd" subsequence has a maximal $\tau$-profile with a single element, the equivalence class of the constant sequence $\phi_k = \phi$.

What we want, then, is to incorporate considerations concerning subsequences into our discussion. To facilitate this, we first make the following easy observation: If $\vec{u}$ is a bounded sequence with a subordinate $\tau$-profile $\mathcal{P}$, then given any subsequence $\vec{u} \circ \zeta$, the $\tau$-profile $\mathcal{Q} := \{[\vec{\phi} \circ \zeta] : [\vec{\phi}] \in \mathcal{P}\}$ is subordinate to $\vec{u} \circ \zeta$. We will say that $\mathcal{Q}$ is the *inherited* $\tau$-profile.

**Definition 2.36.** Given a bounded sequence $\vec{u}$, a subordinate $\tau$-profile $\mathcal{P}$ is said to be *hereditarily maximal* if for any subsequence $\vec{u} \circ \zeta$, the inherited $\tau$-profile is maximal in $\mathfrak{P}[\vec{u} \circ \zeta]$.

The importance of this concept is captured in the following technical observations.

**Lemma 2.37.** *If a bounded sequence $\vec{u}$ in $H^1$ does not $\tau$-weak converge to zero, then there exists a subsequence $\vec{u} \circ \zeta$ admitting a subordinate non-empty $\tau$-profile.*

*Proof of Lemma 2.37.* By Proposition 2.4 we can find a sequence $\vec{y}$ of points such that $\tau_{-\vec{y}}\vec{u}$ does not weakly converge to zero. Noting that every subsequence of $\{\tau_{-y_k} u_k\}$ is a bounded sequence in $H^1$, and so by Banach-Alaoglu has a weakly convergent sub-subsequence, we see that if every weakly convergent subsequence were to weakly converge to 0, then $\tau_{-y_k} u_k \rightharpoonup 0$ necessarily. Hence the contrapositive of this argument implies that at least one weakly convergent subsequence converges to something that is not the zero function. This provides the desired subsequence and its $\tau$-profile. □

Combining Lemma 2.37 with Corollary 2.34 we find:

**Corollary 2.38.** *Given a bounded sequence $\vec{u}$ and a subordinate $\tau$-profile $\mathcal{P}$. We have $\vec{u} - \vec{\sum}\mathcal{P} \overset{\tau}{\rightharpoonup} 0$ if and only if $\mathcal{P}$ is hereditarily maximal.*

The main result in this section, and the main thrust of the concentration compactness argument, is the following *refinement* of the classical Banach-Alaoglu Theorem.

**Theorem 2.39.** *Given any bounded sequence $\vec{u}$, there exists a subsequence $\vec{v} = \vec{u} \circ \zeta$ and a $\tau$-profile $\mathcal{P} \in \mathfrak{P}[\vec{v}]$ that is hereditarily maximal.*

*Proof.* The strategy is a simple refinement/diagonalization argument. At each stage, we keep a subsequence $\vec{v}$ of $\vec{u}$ (initialized to $\vec{u}$), together with a $\tau$-profile $\mathcal{P}$ (initialized to the empty set). If $\mathcal{P}$ is not hereditarily maximal, then we can refine to a new subsequence $\vec{v}'$ for which the inherited $\tau$-profile can be enlarged to $\mathcal{P}'$. Repeat now with $(\vec{v}', \mathcal{P}')$ as the new $(\vec{v}, \mathcal{P})$.

If this process terminates in finitely many steps, we will have obtained a desired pair. The main question is: "what happens when the process continues indefinitely?" Specifically: how can we extract a limiting pair and guarantee that the limiting profile is hereditarily maximal?

To ensure this, we will add new $\tau$-cores to our collection in a way that is roughly "decreasing in size". To help this, let's define the following function: given a bounded sequence of $H^1$ functions $\vec{v}$, let

$$\mathfrak{w}(\vec{v}) := \sup\left\{ |||[\vec{\phi}]||| : \{[\vec{\phi}]\} \in \mathfrak{P}[\vec{w}], \vec{w} \text{ is a subseq. of } \vec{v} \right\}. \tag{2.40}$$

Our algorithm for constructing $\vec{v}$ is as follows:

1. START: set $\vec{v} = \vec{u}$, $\mathcal{P} = \emptyset$, and $n = 0$.
2. IF: $\mathcal{P}$ is hereditarily maximal, return $(\vec{v}, \mathcal{P})$.
   ELSE: increase $n$ by 1 and continue.
3. Since $\mathcal{P}$ is not hereditarily maximal, by Corollary 2.34 we find $\mathfrak{w}(\vec{v} - \vec{\sum}\mathcal{P}) > 0$. There then exists a subsequence of $\vec{v} - \vec{\sum}\mathcal{P}$, indexed by $\zeta$, and a corresponding $\{[\vec{\phi}]\}$ such that

   $$|||[\vec{\phi}]||| \geq \frac{1}{2}\mathfrak{w}(\vec{v} - \vec{\sum}\mathcal{P}).$$

   We can ensure that $\zeta$ acts as the identity for the first $n$ terms.

4. Replace $\vec{v}$ by $\vec{v}\circ\zeta$, and replace $\mathcal{P}$ first by its inherited profile on $\vec{v}\circ\zeta$, and then enlarge it by adding to it $\{[\vec{\phi}]\}$.
5. RETURN to step 2 and repeat.

By the built-in diagonalization, in the case this algorithm runs indefinitely, since the first $n$ terms of all the sequences involved are fixed after the first $n$ steps, we have well-defined limiting sequences. It remains to show that this limiting sequence is hereditarily maximal.

To prove this, notice that the output of $\mathfrak{w}$ is non-increasing from one iteration to the next (using the same argument as the proof of Corollary 2.34). Next, notice that $\limsup \|v_k - (\vec{\sum}\mathcal{P})_v\|_{H^1}$ is also decaying from iteration from the next (as a consequence of (2.24), and the fact that we are passing to subsequences). So by Lemma 2.23, it must be the case that the output value of $\mathfrak{w}$ tends to zero as the number of interations tends to infinity. But this implies then that, for the limiting $\vec{v}$ and $\mathcal{P}$, it holds that $\mathfrak{w}(\vec{v}-\vec{\sum}\mathcal{P}) = 0$, which implies that the limiting $\mathcal{P}$ is hereditarily maximal. □

§2.4 **Optimizer for the Gagliardo-Nirenberg-Sobolev inequality.—** Now we use Theorem 2.39 to solve the minimization problem for the Sobolev inequality $H^1 \hookrightarrow L^q$ on $\mathbb{R}^d$, with $d \geq 3$ and $q \in (2, \frac{2d}{d-2})$. As discussed before, we are interested then in finding a minimizer to the optimization problem

$$\text{minimize } \|u\|_{H^1} \text{ under the constraint } \|u\|_{L^q} = 1.$$

Let $\vec{u}$ be a minimizing sequence. Applying Theorem 2.39 we may assume, after extracting a subsequence, that $\vec{u}$ has a hereditarily maximal subordinate $\tau$-profile $\mathcal{P}$. By Corollary 2.38, this means that $\vec{u}-\vec{\sum}\mathcal{P} \overset{\tau}{\rightharpoonup} 0$, and hence by Theorem 2.11 we have that $\vec{u}-\vec{\sum}\mathcal{P}$ converges to 0 strongly in $L^q$. From this we conclude that $\lim \|\vec{\sum}\mathcal{P}\|_{L^q} = 1$.

On the other hand, by Remark 2.32 we have that $\lim \|\vec{\sum}\mathcal{P}\|_{H^1}^2 = \sum_{[\vec{\phi}]\in\mathcal{P}} \|[\vec{\phi}]\|_{H^1}^2$. Applying Lemma 2.23 we get then $\lim \|\vec{\sum}\mathcal{P}\|_{H^1}^2 \leq \lim \|\vec{u}\|_{H^1}^2$. Since $\vec{u}$ is minimizing, this must be an equality. And hence $\vec{\sum}\mathcal{P}$ is itself a minimizing sequence.

What we will show next, is that in fact, $\mathcal{P}$ must have cardinality 1. If this were the case, let $\phi$ be the shape of an representative of the unique element in $\mathcal{P}$. As the ratio

$$\frac{\|\vec{\sum}\mathcal{P}\|_{H^1}}{\|\vec{\sum}\mathcal{P}\|_{L^q}} = \frac{\|\phi\|_{H^1}}{\|\phi\|_{L^q}}$$

is the constant sequence, for $\vec{\sum}\mathcal{P}$ to be minimizing this would mean that this shape $\phi$ *is itself an optimizer for the Sobolev inequality.*

We will establish finally that $\mathcal{P}$ contains exactly one element through several technical (and mostly elementary) lemmas. For convenience we will denote $C_*$ by the optimum constant for the inequality $\|u\|_{L^q} \leq C_* \|u\|_{H^1}$.

**Lemma 2.41.** *Given $0 < p < q < \infty$, and $X$ a set of non-negative real numbers, then*

$$\left(\sum_{x\in X} x^p\right)^{1/p} \geq \left(\sum_{x\in X} x^q\right)^{1/q},$$

*whenever the sums are well-defined. Furthermore, equality is achieved only when $X$ contains exactly one non-zero element.*

*Proof.* By scaling homogeneity, we can assume that the right side equals 1. This means that each $x \in X$ is $\leq 1$. And hence $x^p \geq x^q$, with equality only when $x = 1$ or 0. Thus $\sum_{x\in X} x^p \geq 1$ with equality only when exactly one element is non-zero (and equals 1). □

**Lemma 2.42.** *Given real numbers $x, y$, and $q \in (1,\infty)$, there exists a universal constant $C_q$ such that*

$$\Big| |x+y|^q - |x|^q - |y|^q \Big| \leq C_q(|x|^{q-1}|y| + |y|^{q-1}|x|).$$

*Proof.* The inequality is trivially true if any one of $x, y, x+y$ is zero, with the constant $C_q = 1$. So we will assume that none of the three vanish. By scaling homogeneity we can assume $y = 1$. So we are down to proving

$$\Big| |1+x|^q - |x|^q - 1 \Big| \leq C_q(|x| + |x|^{q-1}).$$

Consider the function

$$f(x) = \frac{|1+x|^q - |x|^q - 1}{|x| + |x|^{q-1}}.$$

Near $x = 0$, the numerator is differentiable with derivative $q$. The denominator satisfies

$$\lim_{x\to 0^+} \frac{|x| + |x|^{q-1}}{x} = \begin{cases} 1 & q > 2 \\ 2 & q = 2 \\ +\infty & q \in (1,2) \end{cases}$$

$$\lim_{x\to 0^-} \frac{|x| + |x|^{q-1}}{x} = \begin{cases} -1 & q > 2 \\ -2 & q = 2 \\ -\infty & q \in (1,2) \end{cases}$$

So by L'Hôpital's rule, $f$ is bounded near the origin, and continuous away from the origin. Near infinity, we have the numerator satisfies

$$\frac{|1+x|^q}{|x|^q} \approx 1 + \frac{q}{x} \implies |1+x|^q - |x|^q - 1 \approx q|x|^{q-2}x.$$

This shows that $f$ is bounded near infinity. By continuity the result follows. □

**Corollary 2.43.** *Given a finite set of real numbers $X$ and $q \in (1,\infty)$, there exists a constant $C$ dependent on $q$ and the cardinality of $X$ such that*

$$\Big| \Big| \sum_{x\in X} x \Big|^q - \sum_{x\in X} |x|^q \Big| \leq C \sum_{\substack{x,y\in X \\ x\neq y}} |x|^{q-1}|y|.$$

*Proof.* We argue by induction using the previous lemma, and the observation that $|x+y|^{q-1} \leq \tilde{C}(|x|^{q-1} + |y|^{q-1})$ for some $\tilde{C}$ depending on $q$. □

**Lemma 2.44.** *Given any finite subset $\mathcal{P}' \subset \mathcal{P}$, we have that* $\lim \|\vec{\sum}\mathcal{P}'\|_{L^q}^q = \sum_{[\vec{\phi}]\in\mathcal{P}'} \|[\vec{\phi}]\|_{L^q}^q$.

*Proof.* For convenience, index the shapes as $\phi_\alpha$ and progressions as $y_k^\alpha$ for $\alpha \in \{1,\dots,N\}$. Then by Corollary 2.43 we have

$$\left| \Big\| \sum_\alpha \tau_{y_k^\alpha}\phi_\alpha \Big\|_{L^q}^q - \sum_\alpha \|\tau_{y_k^\alpha}\phi_\alpha\|_{L^q}^q \right| \le C \int \sum_{\alpha\neq\beta} |\tau_{y_k^\alpha}\phi_\alpha|^{q-1} |\tau_{y_k^\beta}\phi_\beta| \, dx.$$

The translation invariance of the Lebesgue measure implies

$$= C \int \sum_{\alpha\neq\beta} |\phi_\alpha|^{q-1} |\tau_{y_k^\beta - y_k^\alpha}\phi_\beta| \, dx.$$

When $\alpha \neq \beta$, we have that $|y_k^\beta - y_k^\alpha| \to +\infty$ and hence $|\tau_{y_k^\beta - y_k^\alpha}\phi_\beta| \rightharpoonup 0$. As the sum is finite, this shows that

$$\left| \Big\| \sum_\alpha \tau_{y_k^\alpha}\phi_\alpha \Big\|_{L^q}^q - \sum_\alpha \|\tau_{y_k^\alpha}\phi_\alpha\|_{L^q}^q \right| \to 0. \qquad \square$$

**Corollary 2.45.** $\lim \|\vec{\sum}\mathcal{P}\|_{L^q}^q = \sum_{[\vec{\phi}]\in\mathcal{P}} \|[\vec{\phi}]\|_{L^q}^q$.

*Proof.* By uniform convergence from the Selection Theorem 2.26 and Sobolev's inequality, we have $\vec{\sum}\mathcal{P}$ also converges uniformly in $L^q$. So given any $\epsilon > 0$, we can find a finite subset $\mathcal{P}'$ such that

$$\left| \|\vec{\sum}\mathcal{P}\|_{L^q}^q - \|\vec{\sum}\mathcal{P}'\|_{L^q}^q \right| < \epsilon^q.$$

By Lemma 2.23 and the Sobolev's inequality, we can further ensure, by enlarging $\mathcal{P}'$ if necessary, that

$$\sum_{[\vec{\phi}]\in\mathcal{P}\setminus\mathcal{P}'} \|[\vec{\phi}]\|_{L^q}^2 < \epsilon^2.$$

For $\epsilon < 1$ this further ensures that

$$\sum_{[\vec{\phi}]\in\mathcal{P}\setminus\mathcal{P}'} \|[\vec{\phi}]\|_{L^q}^q < \epsilon^q.$$

By Lemma 2.44 we see that there exists $k_0$ such that for all $k \ge k_0$

$$\left| \Big\| \sum_{[\vec{\phi}]\in\mathcal{P}'} \phi_k \Big\|_{L^q}^q - \sum_{[\vec{\phi}]\in\mathcal{P}'} \|[\vec{\phi}]\|_{L^q}^q \right| < \epsilon^q.$$

Combining everything we find, if $k \ge k_0$,

$$\left| \Big\| \sum_{[\vec{\phi}]\in\mathcal{P}} \phi_k \Big\|_{L^q}^q - \sum_{[\vec{\phi}]\in\mathcal{P}} \|[\vec{\phi}]\|_{L^q}^q \right| < 3\epsilon^q.$$

Since $\epsilon$ is arbitrary this shows our claim. $\square$

Now we can show that $\mathcal{P}$ can contain only one element. Our earlier argument, combined with Corollary 2.45 and the Sobolev inequality shows that

$$1 = \lim \|\vec{\sum}\mathcal{P}\|_{L^q}^q = \sum_{[\vec{\phi}]\in\mathcal{P}} \|[\vec{\phi}]\|_{L^q}^q \leq C_*^q \sum_{[\vec{\phi}]\in\mathcal{P}} \|[\vec{\phi}]\|_{H^1}^q$$

We also had

$$\frac{1}{C_*^2} = \lim \|\vec{\sum}\mathcal{P}\|_{H^1}^2 = \sum_{[\vec{\phi}]\in\mathcal{P}} \|[\vec{\phi}]\|_{H^1}^2.$$

Combining, this shows

$$\left(\sum_{[\vec{\phi}]\in\mathcal{P}} \|[\vec{\phi}]\|_{H^1}^2\right)^{1/2} = \frac{1}{C_*} \leq \left(\sum_{[\vec{\phi}]\in\mathcal{P}} \|[\vec{\phi}]\|_{H^1}^q\right)^{1/q}$$

and in view of Lemma 2.41 this means that $\mathcal{P}$ can only have a single element.

**Remark 2.46.** A similar argument to Remark 1.6 shows the following fact: let $\Omega \subseteq \mathbb{R}^d$ be such that $\Omega^C$ is compact with non-empty interior. Then there exists *no* optimizer to the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^q(\Omega)$.

The reason is this: first, observe that for the same reason explained in Remark 1.6, the optimal constants obey $C_{*,\Omega} \leq C_*$. We first show that they are in fact equal: let $\phi_*$ be the optimizer for the $\mathbb{R}^d$ embedding. Let $\chi$ be a smooth cut-off function that vanishes on $\Omega^C$ and equals 1 slightly away from it. Let $y_k = (k, 0, \ldots, 0)$. Then we have $\chi\tau_{y_k}\phi_* \in H_0^1(\Omega)$, and it is easy to show that as functions on $\mathbb{R}^d$, that $\chi\tau_{y_k}\phi_* - \tau_{y_k}\phi_* \stackrel{\tau}{\rightharpoonup} 0$. Therefore we conclude that

$$\lim \frac{\|\chi\tau_{y_k}\phi_*\|_{H^1}}{\|\chi\tau_{y_k}\phi_*\|_{L^q}} = \frac{1}{C_*}.$$

and this shows that $C_{*,\Omega} \geq C_*$. And hence the two values equal.

But then any optimizer for the $\Omega$ problem will be an optimizer for the whole-space problem, and the same strong maximum principle argument as in Remark 1.6 now shows that therefore optimizers for the $\Omega$ problem cannot exist.

## §3: Non-compactness from Scaling

I will not have time to discuss this in detail. But I want to provide you with a rough account of what happens when studying the existence of the optimizer of the $H_0^1 \hookrightarrow L^q$ embedding where $q = \frac{2d}{d-2}$.

For convenience we will work in $\mathbb{R}^d$ (the domain case is similar, just remove the translations). As discussed already in class, for this particular Sobolev embedding there is another obstruction for compactness, and that is scaling. For convenience we will equip $H_0^1$ with the norm $u \mapsto \|\nabla u\|_{L^2}$. This is not entirely comparable to the full $H^1$ norm, but is non-the-less a norm on the space. Most of the ideas carry over to the full $H^1$ norm case but with some additional technical "complications" that make the ideas less transparent.

Define the scaling operator $\sigma_\lambda u(x) = e^{(1-d/2)\lambda} u(xe^{-\lambda})$. What we find is that

$$\|\nabla \sigma_\lambda u\|_{L^2} = \|\nabla u\|_{L^2} \quad \text{and} \quad \|u\|_{L^q} = \|\sigma_\lambda u\|_q.$$

And so if you take $\phi \in C_0^\infty(\mathbb{R}^d)$ and let $u_k = \sigma_{\pm k}\phi$, you will find that this sequence (with either the $\pm$ sign) will not have any convergent subsequence in $L^q$, even though it is a bounded sequence in $H_0^1$.

This is a *new* phenomenon compared to the previous case where $2 < q < \frac{2d}{d-2}$.

To handle this, in addition to modulate by translations, we need to also modulate by dilations (scaling).

1. We can define the notion of $(\sigma, \tau)$-weak convergence for a sequence $\vec{u}$ in $H_0^1$, by requiring that for any $\varphi \in H_0^1$, that
$$\limsup_{k\to\infty} \sup_{y\in\mathbb{R}^d, \lambda\in\mathbb{R}} \int \nabla(u_k - u)\cdot \nabla(\tau_y \sigma_\lambda \varphi) = 0.$$
2. We can define cores and profiles now, with respect to both $\sigma$ and $\tau$: a $(\sigma,\tau)$-core is a sequence of the form $\tau_{y_k}\sigma_{\lambda_k}\phi$; its progression will be the pair $(\lambda_k, y_k) \in \mathbb{R}^{d+1}$.
3. Equivalence of core is defined the same way, and orthogonality of cores is defined similarly (that $(\lambda_k - \mu_k, y_k - z_k) \to \infty$).
4. Under these definitions, the obvious replacements for Lemma 2.23 and the Selection Theorem 2.26 both still hold. Similarly, our main Theorem 2.39 also is true, with $\tau$-profile replaced by $(\sigma,\tau)$-profile.
5. So, the entire concentration compactness machinery works with almost exactly the same proofs *except* for one thing: and this is Theorem 2.11. The proof of this theorem used as an ingredient the Rellich-Kondrachov Compactness Theorem, which is no longer available. The result, none-the-less, is true: that $u_k \overset{(\sigma,\tau)}{\rightharpoonup} 0 \iff \|u_k\|_{L^q} \to 0$ for $q = \frac{2d}{d-2}$. We record a proof later in this section.
6. Using these results, we see that again we can prove the existence of an optimizer for the Sobolev embedding using pretty much the same argument given above.

To conclude this quick exposition, we record a proof of the replacement of Theorem 2.11.

**Theorem 3.1.** *Let $\vec{u}$ be a bounded sequence of $H_0^1(\mathbb{R}^d)$ functions with the Hilbert space inner produce $\langle u,v\rangle = \int \nabla u\cdot\nabla v$. Then*

$$u_k \overset{(\sigma,\tau)}{\rightharpoonup} 0 \iff \|u_k\|_{L^{\frac{2d}{d-2}}} \to 0.$$

We will write $q = \frac{2d}{d-2}$.

*Proof of* $\Leftarrow$. Let $\varphi \in C_0^\infty(\mathbb{R}^d)$ be arbitrary. Then

$$\langle \tau_{y_k}\sigma_{\lambda_k} u, \varphi\rangle = -\int \tau_{y_k}\sigma_{\lambda_k} u_k \Delta\varphi.$$

Using that $\varphi \in C_0^\infty$, it belongs to $L^p$ where $\frac{1}{p} + \frac{1}{q} = 1$. If $u_k \to 0$ in $L^q$, so do $\tau_{y_k}\sigma_{\lambda_k} u$, and hence the RHS tends to zero. Using that $C_0^\infty$ is dense in $H_0^1$ we obtain our conclusion, after appealing to the version of Proposition 2.4 for the $(\sigma,\tau)$-weak convergence. ☐

*Proof of* $\Rightarrow$. This direction is somewhat more challenging. We wish to follow an argument similar to that of Theorem 2.11 through decomposing our integration, but now we have to also incorporate "scaling" in addition to translations.

The rough idea:

1. First, cut-up each $u_k$ based on its output value so we can write $u_k = \sum_\lambda u_{k,\lambda}$, where
$$u_{k,\lambda}(x) = \begin{cases} u_k(x) & |u_k(x)| \in [e^{(1-d/2)\lambda}, e^{(1-d/2)(\lambda-1)}) \\ 0 & \text{otherwise} \end{cases}$$
2. Then decompose each $u_{k,\lambda}$ into pieces on cubes $Q_{\lambda,\mu}$; here $\mu$ indexes a collection of congruent cubes that cover $\mathbb{R}^d$. But instead of using unit cubes, we now use cubes of side length $e^\lambda$.
3. Key idea: for $\nu \in \mathbb{Z}$, let $\tilde{u}_k = \sigma_\nu u_k$, then the corresponding piece $\tilde{u}_{k,\lambda} = \sigma_\nu u_{k,\lambda-\nu}$. So the scaling allows us to move between "strata". But doing so incurs a penalty from spatial rescaling, so the cubes we use have to be adjusted accordingly. Notice that when function has large value, we use a smaller cube and vice versa.
4. This means that we can follow a similar argument to Theorem 2.11 and identify the cube that contains the most content of $u_k$, and apply a $(\sigma,\tau)$ transformation to move that piece to the unit cube at scale 0 to take advantage of the $(\sigma,\tau)$ weak convergence.

The technical implementation, however, is slightly trickier. This is because just building $u_{k,\lambda}$ as indicated above will force none of the $u_{k,\lambda}$ pieces to be $H^1_0$ functions (due to having jump discontinuity). To account for this, we are going back to an idea you handled on Problem 7 of Problem Set 3.

Step 1: Start with $\mathring{\Xi}$ a smooth function on $\mathbb{R}$, such that it vanishes when $|x| \geq e^{\frac{d}{2}-1}$, and is identically equal to 1 when $|x| \leq 1$. Now, for $\lambda \in \mathbb{Z}$, define
$$\Xi_\lambda(x) = x\left[\mathring{\Xi}(xe^{(d/2-1)\lambda}) - \mathring{\Xi}(xe^{(d/2-1)(\lambda+1)})\right].$$

We have then

- $\sum_\lambda \Xi_\lambda(x) = x$;
- $|\Xi_\lambda(x)| \leq e^{(d/2-1)(1-\lambda)}$.
- there exists a universal constant $C$ such that $|\Xi'_\lambda| \leq C$;
- $\Xi_\lambda(x)\Xi_{\lambda+\mu}(x) = 0$ whenever $\mu \geq 2$.

So this allows us to write $u = \sum_\lambda \Xi_\lambda(u)$. Additionally, we have the scaling relation
$$\Xi_\lambda(\sigma_\nu u) = \sigma_\nu \Xi_{\lambda-\nu} u.$$

From Problem Set 3 #7 we find that each of the $\Xi_\lambda(u)$ terms are individually in $H^1 \cap L^\infty$.

Step 2: Since $\Xi_\lambda(u)$ and $\Xi_{\lambda+\mu}(u)$ have disjoint support for $\mu \geq 2$, we see that
$$\|u\|^q_{L^q} \approx \sum_\lambda \|\Xi_\lambda(u)\|^q_{L^q}.$$

Now, for each $\lambda$, we can smoothly partition $\mathbb{R}^d$ into cubes of side-lengths $e^\lambda$. More precisely: we can find a smooth function $\mathring{\chi}$ such that $\mathring{\chi}$ is supported on the cube $[-1/4, 5/4]^d$ and equals 1 on the cube $[1/4, 3/4]^d$. Furthermore, we require that $\sum_{y\in\mathbb{Z}^d} \tau_y \mathring{\chi} = 1$. Let $\chi_{\lambda,y}$, for $\lambda \in \mathbb{Z}$ and $y \in \mathbb{Z}^d$ be
$$\chi_{\lambda,y}(x) = \tau_y \mathring{\chi}(e^{-\lambda} x).$$

Noting that each $\chi_{\lambda,y}\chi_{\lambda,z} = 0$ if $|y-z|_{\ell^\infty} > 1$, we can further decompose $\Xi_\lambda(u)$ into $\sum_y \chi_{\lambda,y}\Xi_\lambda(u)$, and the fact that the pieces are mostly disjoint allows us to say

$$\|u\|_{L^q}^q \approx \sum_\lambda \sum_y \|\chi_{\lambda,y}\Xi_\lambda(u)\|_{L^q}^q.$$

Similarly, we also have

$$\|\nabla u\|_{L^2}^2 \approx \sum_\lambda \sum_y \|\nabla[\chi_{\lambda,y}\Xi_\lambda(u)]\|_{L^2}^2.$$

By construction if $u \in H_0^1$, then $\chi_{\lambda,y}\Xi_\lambda(u)$ is an $H^1(\mathbb{R}^d)$ function with compact support, since we obtained it from smooth truncation. This means that we can apply the Gagliardo-Nirenberg-Sobolev inequality to find that

$$\|\chi_{\lambda,y}\Xi_\lambda(u)\|_{L^q} \lesssim \|\nabla\chi_{\lambda,y}\Xi_\lambda(u)\|_{L^2}$$

with a universal constant. Using that $q > 2$ we see that

$$\|u\|_{L^q}^q \lesssim \|\nabla u\|_{L^2}^2 \sup_{\lambda,y} \|\chi_{\lambda,y}\Xi_\lambda(u)\|_{L^q}^{q-2}.$$

Therefore, in order to prove that $u_k \to 0$ in $L^q$ under the given hypotheses, *it suffices to show that under the assumption of* $u_k \overset{(\sigma,\tau)}{\rightharpoonup} 0$, *necessarily* $\sup_{\lambda,y}\|\chi_{\lambda,y}\Xi_\lambda(u)\|_{L^q} \to 0$.

Step 3: Arguing similarly to Theorem 2.11 we find a sequence $\lambda_k, y_k$ such that

$$\|\chi_{\lambda_k,y_k}\Xi_{\lambda_k}(u_k)\|_{L^q} \geq \frac{1}{2}\sup \|\chi_{\lambda,y}\Xi_\lambda(u_k)\|_{L^q}.$$

Replacing $\vec{u}$ by $\tau_{-\vec{y}}\sigma_{-\vec{\lambda}}\vec{u}$, we can assume without loss of generality that $y_k = 0$ and $\lambda_k = 0$ for all $k$. Now I claim that

$$\|\chi_{0,0}\Xi_0(u_k)\|_{L^q} \to 0.$$

Using that the functions are uniformly bounded, by the $\Xi$ cut-off, it suffices (using Lebesgue interpolation) to show that

$$\|\chi_{0,0}\Xi_0(u_k)\|_{L^p} \to 0$$

for some $p < q$.

Now, let $\zeta$ be a smooth function with compact support equalling 1 on the support of $\chi_{0,0}$. I claim first that $\zeta u_k \to 0$ in $L^p$. If this were the case, using that $|\Xi_0(u_k)| \leq |u_k|$, we then have $\zeta\Xi_0(u_k) \to 0$ in $L^p$ also. The claim with $\chi_{0,0}$ follows immediately.

To show that $\zeta u_k \to 0$ in $L^p$, observe that since $\|\nabla u_k\|_{L^2}$ is uniformly bounded, that $\|u_k\|_{L^q}$ is also uniformly bounded. Writing

$$\nabla(\zeta u_k) = \nabla\zeta u_k + \zeta\nabla u_k$$

we get

$$\|\nabla(\zeta u_k)\|_{L^2} \leq \|\zeta\|_{L^\infty}\|\nabla u_k\|_{L^2} + \|\nabla\zeta\|_{L^{q'}}\|u_k\|_{L^q}$$

is uniformly bounded. And hence $\zeta u_k \in H_0^1(\Omega')$, where $\Omega'$ is the interior of the support of $\zeta$. Rellich-Kondrachov together with Lemma 1.7 shows then for any $p \in [1,q)$ that $\zeta u_k \to 0$ strongly in $L^p$. □