Euler, Lagrange, Noether, Einstein, and Hilbert

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Overview

1. Lagrangian Field Theory
2. The Euler–Lagrange Equations
3. Noether's Theorem
4. Einstein–Hilbert
5. (time permitting) Going beyond

Part I

Lagrangian Field Theory

Model

1⟩ A "field" is given by a function
Domain: manifold M
Codomain: vector space V

2⟩ **Example (Classical mechanics)**
$M = \mathbb{R}$ (time); $V = \mathbb{R}^3$ (positive of particle).
$\gamma : M \to V$: particle trajectory as a function of time. ♢

3⟩ **Example (Electromagnetism)**
$M = \mathbb{R}^{1,3}$ (Minkowski space-time); $V = \mathbb{R}^3$ (vector potential; temporal gauge).
$A : M \to V$: $\partial_t A$ is electric field, $\nabla \times A$ is magnetic field. ♢

Remark on Generalization

4) In the most general setting, instead of considering maps $M \to V$ we can consider sections of fiber bundles (F, π, M) (or even more generally fibered manifolds).
Instead of bundles one may consider the case where we have maps $M \to N$ where N is also a manifold. (E.g. harmonic maps)
In both cases the extra geometry necessitates introducing some additional language (*linear connection, jet bundle*) which obscures the main topics of today.

5) By taking local coordinates / local trivializations we can reduce to the case of $M \to V$. (Our operations today are all local.)

6) As vector space, we have the canonical trivialization $TV \cong V \times V$, simplifying the picture.

Lagrangian Field Theory

7) For convenience, fix volume form $dvol$ on M

8) Action Principle: physical solutions are (formal) critical points of an action (Lagrangian).

9) Physics: action should depend on Kinetic and Potential energies; $\implies$ depend on the value of the function and its first derivative.

10) Configuration Space: "all possible pointwise configurations of the field"
- First derivative: a section of $T^*M \otimes V$ (V-valued one-form)
- Field itself: a function $M \to V$
- Configuration space: $(T^*M \otimes V) \times V$.

(Geometrically the configuration space should be the first jet bundle $j^1(M, V)$; in our simplified setting the above is canonically isomorphic.)

Lagrangian Field Theory

11⟩ Lagrangian Density: $L : (T^*M \otimes V) \times V \to \mathbb{R}$.

12⟩ Action: $\phi : M \to V$

$$S[\phi] = \int L(\underbrace{p, d\phi|_p}_{\text{Section of } T^*M\otimes V}, \phi_p)\, dvol$$

13⟩ **Remark**

An alternative geometric formulation without fixing a *dvol* is to let L be a bundle map from $(T^*M \otimes V) \times V \to \Lambda^{\text{top}} T^*M$, so that the volume form is incorporated as part of L. This has little practical effect. ♢

Part II

Euler–Lagrange Equations

"Formal" Action

14⟩ The action is often referred to as "formal", as for actual solutions it is generally the case that the integral $\int L\, dvol$ does *not* converge, due to M being often non-compact.

15⟩ To formulate a variation problem, consider a one-parameter family of *compactly supported perturbations*, and perform the integration only on a compact set. The Euler–Lagrange Equations are still well-defined even though the action is not.

Variation

16⟩ Let $s \mapsto \phi(s;-)$ be a one-parameter family of fields, that agree outside a compact set K.

17⟩ Since ϕ take values in a vector space, $\dot{\phi}(s;-) : M \to V$ is well-defined, as is its differential $d\dot{\phi}$.

18⟩ $d\phi(s;-)$ is a one parameter family of sections of $T^*M \otimes V$; its s-derivative is equal to $d\dot{\phi}$.

19⟩ $\phi(0;-)$ is a formal critical point of $\mathcal{S}$ means

$$\frac{d}{ds}\mathcal{S}[\phi(s;-)]\Big|_{s=0} = 0.$$

Variation

20⟩ Chain rule:

$$\frac{d}{ds}L(p, d\phi(s;p), \phi(s;p)) = \frac{\partial}{\partial\phi}L(p, d\phi(s;p), \phi(s;p)) \cdot \dot{\phi}(s;p) + \frac{\partial}{\partial(d\phi)}L(p, d\phi(s;p), \phi(s;p)) \cdot d\dot{\phi}(s;p).$$

21⟩ The partials on the right are well-defined, since fixing a base point p, the fiber of the configuration space $(T^*M \otimes V) \times V$ is the vector space $T^*_pM \otimes V) \times V$, thanks to our simplifying assumptions.

Object types

$$\frac{d}{ds}L(p, d\phi(s;p), \phi(s;p)) = \frac{\partial}{\partial\phi}L(p, d\phi(s;p), \phi(s;p)) \cdot \dot{\phi}(s;p) + \frac{\partial}{\partial(d\phi)}L(p, d\phi(s;p), \phi(s;p)) \cdot d\dot{\phi}(s;p).$$

22⟩ $\frac{\partial}{\partial(d\phi)}L(p, d\phi(s;p), \phi(s;p))$ can be acted on $d\dot{\phi}(s;p)$ to get a scalar, so at p is in $T_pM \otimes V^*$.

In other words: $\frac{\partial}{\partial(d\phi)}L$ is a V^*-valued vector field.

23⟩ Similarly $\frac{\partial}{\partial\phi}L(p, d\phi(s;p), \phi(s;p))$ acts on $\dot{\phi}(s;p)$, so belongs to V^*.

Lie Differentiation

24⟩ Let X be a vector field, and f a function, then

$$X(f)\,dvol = \mathcal{L}_X(f)\,dvol = \mathcal{L}_X(f\,dvol) - f\,\mathcal{L}_X(dvol)$$

25⟩ $\mathcal{L}_X(f\,dvol) = d(f\,\iota_X dvol)$ is exact

26⟩ $f\,\mathcal{L}_X(dvol) = f\,\mathrm{div}(X)\,dvol$ by definition of divergence

27⟩ Extends (by linearity) also to V, V^* valued functions and vector fields:

$$\left(\frac{\partial}{\partial(d\phi)}L\right)\cdot d\dot{\phi}\,dvol = d\left(\dot{\phi}\cdot\iota_{\partial_{d\phi}L}dvol\right) - \dot{\phi}\,\mathrm{div}\left(\frac{\partial}{\partial(d\phi)}L\right)dvol$$

Back to the Critical Point

$$\frac{d}{ds}\mathcal{S}[\phi(s;-)] = \int_K d\big(\dot{\phi} \cdot \iota_{\partial_{d\phi}L} dvol\big) + \int_K \Big(\frac{\partial}{\partial\phi}L - \operatorname{div}\big(\frac{\partial}{\partial(d\phi)}L\big)\Big) \cdot \dot{\phi}\, dvol$$

28⟩ Recall: K is compact, contains support of $\dot{\phi}$.

- Integral converges.
- First integral vanishes (Stokes Theorem).

Euler–Lagrange Equations

29⟩ Suppose ψ is such that for all 1-parameter variations $\phi(s; p)$ with $\phi(0; p) = \psi(p)$ we have $\frac{d}{ds} S[\phi(s; -)]\Big|_0 = 0$.

30⟩ $\implies$ For all $\dot{\phi} : M \to V$ with compact support

$$0 = \int \Big(\frac{\partial}{\partial \phi} L(p, d\psi(p), \psi(p)) - \operatorname{div}\Big(\frac{\partial}{\partial (d\phi)} L(p, d\psi(p), \psi(p)) \Big) \Big) \cdot \dot{\phi} \, dvol$$

$\implies$ (pointwise everywhere)

$$0 = \frac{\partial}{\partial \phi} L(p, d\psi(p), \psi(p)) - \operatorname{div}\Big(\frac{\partial}{\partial (d\phi)} L(p, d\psi(p), \psi(p)) \Big)$$

This V^*-valued system form the Euler–Lagrange equations.

Example: scalar field

31⟩ Set $V = \mathbb{R}^d$ with inner product $\langle -, - \rangle$.

32⟩ Set $M = \mathbb{R} \times \mathbb{R}^n = \{(t, \vec{x})\}$, standard volume form $dt \wedge dx_1 \wedge \cdots \wedge dx_n$.

33⟩ Scalar field Langrangian density:

$$L((t, \vec{x}), d\phi, \phi) = -\langle \partial_t \phi, \partial_t \phi \rangle + \sum_{i=1}^{n} \langle \partial_{x_i} \phi, \partial_{x_i} \phi \rangle.$$

34⟩ $\dfrac{\partial}{\partial \phi} L = 0$; and

$$\frac{\partial}{\partial (d\phi)} L \cdot d\dot{\phi} = -2 \langle \partial_t \phi, \partial_t \dot{\phi} \rangle + 2 \sum_{i=1}^{n} \langle \partial_{x_i} \phi, \partial_{x_i} \dot{\phi} \rangle.$$

Example: scalar field

35⟩ Euler–Lagrange Equations (after identifying $V = V^*$ using inner product)

$$\frac{\partial}{\partial \phi} L - \operatorname{div}\Big(\frac{\partial}{\partial (d\phi)} L \Big) = 2\partial^2_{tt}\psi - 2\triangle\psi = 0$$

(linear wave equation)

Summary

36⟩ Action principle: look for critical points of an action functional

37⟩ No geometric structure required on the domain manifold M

38⟩ Critical points $\iff$ solve a geometric PDE, the Euler–Lagrange equation

Part III

Noether's Theorem

39⟩ **Theorem (Noether; imprecise version)**
If the action $\mathcal{S}$ has a continuous symmetry, then every critical point *of $\mathcal{S}$ (solution to Euler–Lagrange equations) has a corresponding conservation law.* ∎

- What is a symmetry?
- What is a conservation law?

40⟩ **Definition**
A diffeomorphism Φ *is a* symmetry *of the action* $\mathcal{S}$ *if*

$$\int_{\Phi(\Omega)} L(p, d\phi(p), \phi(p))\, dvol = \int_{\Omega} L(p, \Phi^*(d\phi)(p), \Phi^*(\phi)(p))\, dvol$$

for every open $\Omega \Subset M$*, and every* $\phi : M \to V$*.* ♢

41⟩ Key point: Φ acts on the domain and ϕ, but not on L or $dvol$.

42⟩ One can (dually) define a symmetry by holding the domain and ϕ fixed, but moving $L\, dvol$: the latter is a bundle map from the configuration space (as a vector bundle over M) to $\Lambda^{top} T^* M$. A diffeomorphism $\Phi : M \to M$ induces a morphism of such maps; and we can define "symmetry of the action" as diffeomorphisms that leave this bundle map invariant.

Infinitesimal Consequence

43⟩ Now suppose Φ_s is a one parameter family of symmetries of the action.

44⟩ $\frac{\partial}{\partial s}\Phi_s\Big|_{s=0}$ is a vector field on M, call it X.

45⟩ Taking the s derivative at 0 of

$$\int_{\Phi_s(\Omega)} L(p, d\phi(p), \phi(p))\ dvol = \int_{\Omega} L(p, \Phi_s^*(d\phi)(p), \Phi_s^*(\phi)(p))\ dvol$$

yields

$$\int_{\partial\Omega} L(p, d\phi(p), \phi(p))\ \iota_X dvol = \int_{\Omega} \frac{d}{ds} L(p, \Phi_s^*(d\phi)(p), \Phi_s^*(\phi)(p))\ dvol\Big|_{s=0}.$$

Infinitesimal Consequence

46⟩ $\Phi_s^*(\phi)$ is a one parameter family of fields, can apply slide 13

$$\int_{\partial\Omega} L\, \iota_X dvol = \int_\Omega d\Big(X(\phi)\cdot \iota_{\partial_{d\phi}L} dvol\Big) + \int_\Omega \Big(\frac{\partial}{\partial\phi}L - \mathrm{div}\big(\frac{\partial}{\partial(d\phi)}L\big)\Big)\cdot X(\phi)\, dvol.$$

47⟩ If ϕ is a critical point, then apply Euler–Lagrange to drop final integral.

48⟩ Stokes' Theorem $\implies$

$$0 = \int_\Omega d\Big(L\, \iota_X dvol - X(\phi)\cdot \iota_{\partial_{d\phi}L} dvol\Big).$$

Holds for all Ω so integrand vanishes pointwise!

49⟩ **Theorem (Noether)**
Suppose Φ_s is a one-parameter family of symmetries of $\mathcal{S}$, with $\frac{\partial}{\partial s}\Phi_s\big|_{s=0} = X$. Given ϕ a critical point of $\mathcal{S}$. Then the vector field

$$^{(X)}j[\phi] := L(p, d\phi, \phi)X - \Big(\frac{\partial}{\partial(d\phi)}L(p, d\phi, \phi)\Big) \cdot X\phi$$

is divergence free. We call $^{(X)}j[\phi]$ *the* Noether current *for the vector field X and the solution ϕ.* ■

Canonical Stress-Energy

50) The formula $X \mapsto {}^{(X)}j[\phi]$ is clearly *tensorial*, define the canonical stress-energy for a solution ϕ to be the type $(1,1)$ tensor field $T_{\text{can}}[\phi]$ given by

$$T_{\text{can}}[\phi](X) := L(p, d\phi, \phi)X - \Big(\frac{\partial}{\partial(d\phi)}L(p, d\phi, \phi)\Big) \cdot X\phi.$$

51) The canonical stress–energy tensor is well-defined for any field ϕ. Noether's theorem says that when ϕ is a solution and when X generates a symmetry, we have $T_{\text{can}}[\phi](X)$ is conserved.

Part IV

Einstein–Hilbert

Dependence on Geometry

52⟩ General relativity: domain is a Lorentzian manifold (M, g)
Lagrangian depends on metric g and the volume form is the metric volume form.

53⟩ Physical assumption: laws of physics independent of space-time location (diffeomorphism invariance)
Lagrangian satisfies, for all diffeomorphism Φ we have

$$L(\Phi^* g, \Phi^*(d\phi), \Phi^*\phi)\Phi^* dvol = \Phi^*\big(L(g, d\phi, \phi)\ dvol\big).$$

Symmetry

54⟩ Φ_s is a one-parameter family of symmetries

$$\int_{\Phi_s(\Omega)} L(g, d\phi, \phi)\ dvol = \int_\Omega L(g, \Phi_s^*(d\phi), \Phi_s^*(\phi))\ dvol$$

55⟩ Apply diffeomorphism invariance

$$\int_\Omega L(\Phi_s^* g, \Phi_s^*(d\phi), \Phi_s^* \phi)\ \Phi_s^*(dvol) = \int_\Omega L(g, \Phi_s^*(d\phi), \Phi_s^*(\phi))\ dvol.$$

56⟩ *Sufficient* (but not necessary condition) is Φ_s acts as isometry of (M, g).

57⟩ Another possibility: Φ_s acts as conformal isometry, and the scalar rescaling factors cancel out. E.g. electromagnetism in 1+3D.

Infinitesimal consequence

58⟩ Take s derivative of

$$L(\Phi_s^* g, d\phi, \phi)\ \Phi_s^*(dvol) = L(g, d\phi, \phi)\ dvol$$

and evaluating at $s = 0$ gets

$$\frac{\partial}{\partial g} L(g, d\phi, \phi) \cdot \partial_s(\Phi_s^* g)\Big|_{s=0} dvol + L(g, d\phi, \phi)\, \partial_s(\Phi_s^*(dvol))\Big|_{s=0} = 0.$$

(Here $\frac{\partial}{\partial g} L$ is a $(2, 0)$ tensor field on M.)

59⟩ Standard computation using Jacobi's identity

$$\partial_s(\det A) = \det(A)\mathrm{tr}(A^{-1}\partial_s A)$$

yields

$$\partial_s(\Phi_s^*(dvol))\Big|_{s=0} = \frac{1}{2}\mathrm{tr}(g^{-1}\partial_s \Phi_s^*(g))\Big|_{s=0} dvol.$$

Infinitesimal consequence

60⟩ Summarize: if $\partial_s \Phi_s\big|_{s=0} = X$, then symmetry implies

$$\Big(\underbrace{\frac{\partial}{\partial g} L(g, d\phi, \phi) + \frac{1}{2} L(g, d\phi, \phi) g^{-1}}_{T_{\text{EH}}[g,\phi]} \Big) \cdot \mathcal{L}_X g = 0.$$

61⟩ The type-$(2,0)$ *Einstein–Hilbert* stress-energy tensor $T_{EH}[g,\phi]$ is *divergence free* if g is a critical point of the Einstein–Hilbert functional

$$\mathcal{S}_{\text{EH}} = \int R + L(g, d\phi, \phi)\, dvol$$

- R is scalar curvature of g
- ϕ can be any fixed field

Einstein–Hilbert current

62⟩ Since $T_{\mathrm{EH}}[g,\phi]$ is symmetric, if g is critical point

$$T_{\mathrm{EH}}[g,\phi]\cdot\mathcal{L}_X g = 2\mathrm{div}\Big(T_{EH}[g,\phi]\cdot X^\flat\Big)$$

so the *Einstein–Hilbert current* vector field $T_{\mathrm{EH}}[g,\phi]\cdot X^\flat$ is divergence free: provides another conservation law.

Part V

Noether versus Einstein–Hilbert

Stress-Energies and Symmetries

63⟩ General relativity setting: (M, g, ϕ)
Total action

$$S = \int R + L(g, d\phi, \phi)\, dvol$$

64⟩ Two stress-energy tensors:

- Type-$(1,1)$ canonical stress T_{can} / variation of ϕ
- Type-$(2,0)$ Einstein–Hilbert stress T_{EH} / variation of g

65⟩ Assume vector field X generates isometries.
g **arbitrary,** ϕ **critical:** $T_{can} \cdot X$ is divergence free
g **critical,** ϕ **arbitrary:** $T_{\text{EH}} \cdot X^\flat$ is divergence free

Comparison

66⟩ *When both g and ϕ critical, are the two related?*

$$T_{\text{can}}(X) = LX - (\frac{\partial}{\partial(d\phi)}L) \cdot X\phi$$

$$2T_{\text{EH}} \cdot X^\flat = LX + 2(\frac{\partial}{\partial g}L) \cdot X^\flat$$

67⟩ The two are equal if

$$2\frac{\partial}{\partial g}L + \frac{\partial}{\partial(d\phi)}L \otimes \nabla\phi = 0.$$

68⟩ Sufficient condition: $L = L(g^{-1}(d\phi, d\phi), \phi)$
Covers many common field theories

Part VI

Applications to Hyperbolic PDEs

Finite speed of propagation

69⟩ Suppose X is a symmetry, and Ω is a set such that $\partial\Omega = \Gamma_- + \Gamma_+$, such that the integral

$$\int_{\Gamma_\pm} \iota^{(X)} j \, dvol$$

is *definite* (only vanishes when $\phi \equiv 0$).
Then $\phi|_{\Gamma_-} = 0$ implies $\phi|_{\Gamma_+} = 0$.

70⟩ Such hypotheses are satisfied when the Euler–Lagrange equations are *hyperbolic*, and this is the prototype for "finite speed of propagation"

71⟩ Observation going back to Leray:
Same can be said even if X is not a symmetry (using Gronwall-type arguments).

Energy Estimates

72⟩ As the stress energy tensors depend only on $d\phi$ and ϕ, its coercivity can at most control ϕ in $W^{1,p}(\Gamma_\pm)$.

73⟩ How to provide *higher order $W^{k,p}$ control*?
⟶ required for proving existence of solutions to the initial value problem.

74⟩ "Reverse engineer" the connection between the Euler–Lagrange equation and the canonical stress tensor.
Can be made precise by linearization around the solution.

Simplest case

75⟩ Consider $M = \mathbb{R}^n$. Suppose Lagrangian

$$L = L((p, d\phi) = h^{\alpha\beta}_{AB}(p)\partial_\alpha\phi^A\partial_\beta\phi^B$$

then Euler–Lagrange equation has principal part

$$h^{\alpha\beta}_{AB}(p)\partial^2_{\alpha\beta}\phi^B + \ldots = 0.$$

And the canonical stress tensor appears as

$$T_{\text{can}}[d\phi]^\mu_\nu = h^{\alpha\beta}_{AB}\partial_\alpha\phi^A\partial_\beta\phi^B\delta^\mu_\nu - h^{\alpha\mu}_{AB}\partial_\alpha\phi^A\partial_\nu\phi^B.$$

76⟩ Call the mapping from the coefficients

$$(h^{\alpha\beta}_{AB}, d\phi) \to T_{\text{can}}[d\phi]$$

the *Noether transform*.

Noether transform

77) *We can apply the Noether transform to PDEs that are not necessarily Lagrangian.* If ϕ solves

$$h^{\alpha\beta}_{AB}(p)\partial^2_{\alpha\beta}\phi^B + \ldots = 0$$

then forming T_{can} from the Noether transform gives

$$\text{div} T_{\text{can}} = O(\phi, d\phi)$$

which allows us to use Leray's argument.

78) Christodoulou calls those $h^{\alpha\beta}_{AB}$ that has a Noether transform with coercivity properties (relative to a vector field X and a hypersurface Σ) *regularly hyperbolic*.

Higher order versus first order energies

79⟩ Assume $L = L(p, d\phi)$. Euler–Lagrange is

$$\operatorname{div}\left(\frac{\partial}{\partial(d\phi)} L(x, d\phi)\right) = 0.$$

Take a derivative, we find

$$\operatorname{div}\left(\frac{\partial^2}{\partial(d\phi)^2} L(x, d\phi)\partial(d\phi)\right) + \ldots = 0$$

so $\partial\phi$ solves a second order PDE with

$$h^{\alpha\beta}_{AB}(x, d\phi) = \frac{\partial^2}{\partial(\partial_\alpha\phi^A)\partial(\partial_\beta\phi^B)} L(x, d\phi).$$

Higher order versus first order energies

80⟩ Key observation: in general, the canonical stress energy of L is *not* equal to the Noether transform of $(h^{\alpha\beta}_{AB}, d\phi)$.

81⟩ (Equal when L is quadratic in $d\phi$.)

82⟩ The canonical stress energy is "better behaved" algebraically because it captures special cancellations from the Lagrangian structure at the lowest derivative level.

83⟩ For understanding the existence theory of hyperbolic PDEs, these cancellations are "too nice" and "non generic". The energy derived from taking the Noether transform of the principal symbol should be used instead.

Thank you for your attention!