GEODESIC CONGRUENCES

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The notion of a *congruence (of curves)* is frequently used in mathematical relativity; in general, a congruence is nothing more than a foliation of a manifold (or a subset thereof) by one-dimensional leaves.

Definition 1. Given a manifold $\mathcal{M}$. A *congruence of curves* is a set $\mathcal{C}$ of curves (one-dimensional submanifold of M) such that for every point $p \in M$, there exists a unique $\gamma_p \in \Gamma$ where $p \in \gamma_p$.

Assuming that the curves are regular, the congruence defines a distinguished subspace of the tangent bundle at each point, given by $\operatorname{span}\{\gamma_p'\}$; conversely, one way of generating a congruence of curves is to start with a non-vanishing vector field v on M, then the set of all maximally extended integral curves of v form a congruence, by Picard's theorem. Below we shall always assume that our congruence can be thus generated by a *smooth* vector field.

In mathematical relativity the notion of congruences are frequently used to represent *world-lines* of a family of observers, or a family of "test particles". Under reasonable assumptions on the underlying space-time, such world-lines should not be *closed* or *self-intersecting*, and hence we can assume that the curves in the congruence are all diffeomorphic to $\mathbb{R}$. In order for the geometry of the congruence to capture the intrinsic geometry of the space-time, and not that of external influences, we typically ask that these world-lines correspond to "free-falling observers". In other words, we are particularly interested in *congruences of causal geodesic curves*. In this note, we develop some language and tools to discuss the geometry of these congruences.

Index of Common Notations

$\mathcal{M}$ — the space-time manifold

$\mathsf{T}\mathcal{M}$ — its tangent bundle

n — the space-time dimension

$\mathrm{Fl}_v(s,p)$ — the flow map for the vector field v starting at the point p with "time" parameter s (see §1.1)

$\mathcal{C}$ — a congruence of curves on $\mathcal{M}$

$\mathcal{V}$ — the subbundle of $\mathsf{T}\mathcal{M}$ given by the kernel of the projection $\mathsf{T}\mathcal{M} \to \mathsf{T}\mathcal{C}$

$\mathcal{H}$ — pullback of $\mathsf{T}\mathcal{C}$ to be over $\mathcal{M}$, an $(n-1)$-dimensional vector bundle

∇ — a torsion-free linear connection

Riem — the Riemann curvature of a connection, with sign convention $\mathrm{Riem}(X,Y)Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z$

Ric — the Ricci curvature of a connection

θ — the expansion of a geodesic congruence

κ — the acceleration of a geodesic vector field, i.e. $\nabla_v v = \kappa v$

1. Differential Geometry

1.1. **Assumptions.** We shall assume our space-time is given by an n-dimensional manifold $\mathcal{M}$. It is equipped with a congruence, which we will assume to be generated by a nowhere-vanishing (smooth) vector field v. We will use Fl_v to denote the flow map for the vector field v; that is, given $p \in \mathcal{M}$, consider the ordinary differential equation

$$\gamma'(s) = v \circ \gamma(s),$$
$$\gamma(0) = p.$$

Let $I_p \subseteq \mathbb{R}$ be the maximal interval of existence for this equation, then for $s \in I_p$, we define $\mathrm{Fl}_v(s,p) = \gamma(s)$, this defines a function Fl_v with domain

$$\{(s,p) \in \mathbb{R} \times \mathcal{M} : s \in I_p\}$$

and codomain $\mathcal{M}$. The smooth-dependence on initial data implies that Fl_v is a smooth map. Our congruence can be given by

$$\mathcal{C} := \{\mathrm{Fl}_v(I_p, p) : p \in \mathcal{M}\}. \tag{2}$$

We shall further assume that the curves in these congruence are all diffeomorphic to $\mathbb{R}$, so that given $p, q \in \mathcal{M}$ such that the curves $\mathrm{Fl}_v(I_p, p) = \mathrm{Fl}_v(I_q, q)$, there exists a unique value s such that $\mathrm{Fl}_v(s,p) = q$ and $\mathrm{Fl}_v(-s,q) = p$.

1.2. **Fibration.** We may regard $\mathcal{C}$ as providing an equivalence relation between space-time events. We would like to use this to decompose our space-time.

Definition 3. By a *transverse section* of the congruence $\mathcal{C}$, we mean a codimension 1 submanifold Σ of $\mathcal{M}$ such that each $\gamma \in \mathcal{C}$ intersects Σ *exactly once* and *transversely*.

Given Σ and Σ' two (smooth) transverse sections, by our assumptions there exists a function $f : \Sigma \to \mathbb{R}$ such that the mapping

$$\Sigma \ni p \mapsto \mathrm{Fl}_v(f(p), p)$$

is a bijection between Σ and Σ'. By the implicit function theorem we can conclude that f is smooth, and hence Σ and Σ' are diffeomorphic. We can therefore use this to define a smooth structure on the set $\mathcal{C}$. With this smooth structure, the projection map

$$\mathcal{M} \ni p \mapsto \mathrm{Fl}_v(I_p, p) \in \mathcal{C} \tag{4}$$

is a smooth submersion, and we can regard $\mathcal{M}$ as a fibred manifold over $\mathcal{C}$ with one-dimensional fibers. The converse operation also holds: given an $(n-1)$-dimensional manifold B, let $\pi : \mathcal{M} \to B$ be a submersion, with each fiber $\pi^{-1}(b)$ diffeomorphic to $\mathbb{R}$. Then the fibers define a congruence of curves automatically, and the resulting $\mathcal{C}$ is diffeomorphic to B. Therefore we shall use these viewpoints interchangeably.

The congruence defines/is defined by a one dimensional distribution v in $\mathsf{T}\mathcal{M}$; for convenience we record the associated subbundle as $\mathcal{V}$. At each $p \in \mathcal{M}$, the vector v_p defines an equivalent relation: two vectors ξ, ζ are considered equivalent if they differ by a multiple of v_p. This defines an $(n-1)$-dimensional vector bundle $\mathcal{H}$ over $\mathcal{M}$; in other words $\mathcal{H}$ is the quotient $\mathsf{T}\mathcal{M}/\mathcal{V}$. An alternative description of this bundle is that it is the pull back of the tangent bundle $\mathsf{T}\mathcal{C}$ by the projection map given in (4).

2. Geometry of geodesic congruences

Our goal is to study the geometry of the aforementioned pullback bundle $\mathcal{H}$. Roughly speaking, how the geometry of this bundle evolves along a curve $\gamma \in \mathcal{C}$ should reflect how nearby curves to γ compare against γ. In the general setting, however, there is too many degrees of freedom. For arbitrary choices v, there are no clear connections between the geometry of $\mathcal{H}$ (in fact, there is no clear way to even define a geometry for $\mathcal{H}$) and the space-time geometry of $\mathcal{M}$ as a pseudo-Riemannian manifold. However, we expect that when $\mathcal{C}$ represent observers in free-fall, that the geometry of $\mathcal{M}$ can be suitably captured.

2.1. **Geodesic congruences.** We will start more generally: in this subsection we will assume that $\mathcal{M}$ is merely equipped with a linear, torsion-free connection ∇ on its tangent bundle. By way of setting a sign convention, we recall the definition of the Riemann and Ricci curvatures for linear connections.

Definition 5. Given a linear torsion-free connection ∇ on the tangent bundle $\mathsf{T}\mathcal{M}$, its *Riemann curvature* is a section of $\mathsf{T}^{1,3}\mathcal{M}$ given by

$$\mathrm{Riem}(X,Y)Z = \nabla_{[X,Y]}Z - \nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z.$$

The corresponding *Ricci curvature* is the partial trace

$$\mathrm{Ric}(X,Z) = \mathrm{tr}(Y \mapsto \mathrm{Riem}(X,Y)Z).$$

We shall further assume that the vector field v is *geodesic*, that is $\nabla_v v \propto v$. Consider now the $\mathsf{T}^{1,1}\mathcal{M}$ tensor ∇v, which we can regard as a linear transformation from $\mathsf{T}_p\mathcal{M}$ to itself at every $p \in \mathcal{M}$. Fix $\xi \in T_p\mathcal{M}$; for any $\lambda \in \mathbb{R}$, we can compute

$$\nabla_{\xi + \lambda v_p} v = \nabla_\xi v + \lambda \underbrace{\nabla_{v_p} v}_{\propto v}. \tag{6}$$

Hence ∇v induces a bundle map $\mathcal{H} \to \mathcal{H}$, sending the equivalence class of $\xi \in T_p\mathcal{M}$ to the equivalence class of $\nabla_\xi v$.

Additionally, if we had selected a different generating vector field $\tilde{v}$ for $\mathcal{C}$ (so that there is a non-vanishing function $\phi : \mathcal{M} \to \mathbb{R}$ such that $\tilde{v} = \phi v$), we find that

$$\nabla_\xi \tilde{v} = \xi(\phi) v + \phi \nabla_\xi v. \tag{7}$$

The first factor being in the direction of v is discarded when we take the equivalence class to reduce to an element of $\mathcal{H}$.

Theorem 8. *Given $\mathcal{M}$ a manifold, ∇ a linear, torsion-free connection, and $\mathcal{C}$ a geodesic congruence, there exists a section B of $\mathcal{V}^* \otimes \mathcal{H}^* \otimes \mathcal{H}$, such that given an equivalence class $[\xi] \in \mathcal{H}_p$ with $\xi \in \mathsf{T}_p\mathcal{M}$ a representative, and v a section of $\mathcal{V}$, the vector $\nabla_\xi v \in \mathsf{T}_p\mathcal{M}$ belongs to the equivalence class $B(v,[\xi]) \in \mathcal{H}$.*

Definition 9. The *expansion* of the geodesic congruence $\mathcal{C}$ is the section θ of $\mathcal{V}^*$ given by the trace (between the $\mathcal{H}^*$ and $\mathcal{H}$ factors) of B from the previous theorem.

To compute θ, we can choose a moving frame $\{v = e_1, e_2, \dots, e_n\}$ on $\mathcal{M}$, with $e_2, \dots, e_n$ serving as representatives of $\mathcal{H}$; denote also by $\{f^1, \dots, f^n\}$ the dual frame. Then

$$\theta(v) = \sum_{j=2}^n f^j(\nabla_{e_j} v) = \sum_{j=1}^n f^j(\nabla_{e_j} v) - f^1(\nabla_v v).$$

In particular,

Proposition 10. *If v is affinely parametrized (so $\nabla_v v = 0$), then $\theta(v) = \mathrm{tr}(\nabla v)$.*

How does the expansion evolve? For this it is again most convenient to consider the value with respect to an affinely parametrized v. Using the previous proposition, we can compute, given a frame $\{e_1, \dots, e_n\}$ and dual frame $\{f^1, \dots, f^n\}$:

$$\begin{aligned} v(\theta(v)) = \nabla_v \mathrm{tr}(\nabla v) = \mathrm{tr}(\nabla_v \nabla v) &= \sum f^i(\nabla_v \nabla_{e_i} v - \nabla_{\nabla_v e_i} v) \\ &= \sum f^i(\nabla_{e_i} \nabla_v v - \nabla_{\nabla_{e_i} v} v + \mathrm{Riem}(e_i, v)v) = -\mathrm{tr}(\nabla v \cdot \nabla v) - \mathrm{Ric}(v,v). \end{aligned}$$

We have therefore proven

Theorem 11 (Raychaudhuri's equation)**.** *Given $\mathcal{M}$ a manifold, ∇ a linear, torsion-free connection, and $\mathcal{C}$ a geodesic congruence, for v an affinely parametrized vector field generating $\mathcal{C}$, we find its corresponding expansion satisfies*

$$v(\theta(v)) = -\mathrm{tr}(\nabla v \cdot \nabla v) - \mathrm{Ric}(v,v).$$

2.2. **Gauge fixing.** While we have $\mathcal{H}_p = \mathsf{T}_p\mathcal{M}/\mathcal{V}_p$, there is no a priori preferred way of identifying $\mathcal{H}_p$ with a subspace of $\mathsf{T}_p\mathcal{M}$. This, however, can be achieved with the prescription of a one-form μ on $\mathcal{M}$ with $\mu(v) \neq 0$, where v is a generator of $\mathcal{C}$. The requirement that $\mu(v) \neq 0$ guarantees that $\ker(\mu)$ is transverse to $\mathcal{V}$. With both the choice of v and μ, we can define a projection operator to $\ker(\mu)$ by

$$\zeta \mapsto \zeta - \frac{\mu(x)}{\mu(v)} v. \tag{12}$$

The tensor B of Theorem 8 can then be realized as a section of $\mathcal{V}^* \otimes \mathsf{T}^*\mathcal{M} \otimes \mathsf{T}\mathcal{M}$ given by

$$B(v, \zeta) = \nabla_\zeta v - \frac{\mu(\nabla_\zeta v)}{\mu(v)} v. \tag{13}$$

Noting that for any geodesic vector field v, we have $B(v,v) = 0$, we see that with this realization, we can set

$$\theta(v) = \mathrm{tr} B(v, -) = \mathrm{tr}(\nabla v) - \frac{\mu(\nabla_v v)}{\mu(v)}. \tag{14}$$

Note that this definition is *independent of the choice of μ, provided that* $\mu(v) \neq 0$. In particular, if v satisfies $\nabla_v v = \kappa v$, then $\mu(\nabla_v v)/\mu(v) = \kappa$ for any μ. And so an alternative formula for the expansion, that works for any representative geodesic vector field, is

$$\theta(v) = \mathrm{tr}(\nabla v) - \kappa, \qquad \text{where } \nabla_v v = \kappa v. \tag{15}$$

We can check tensoriality directly from (14).

$$\theta(\phi v) = \phi \mathrm{tr}(\nabla v) + v(\phi) - \phi \frac{\mu(\nabla_v v)}{\mu(v)} - v(\phi) = \phi \theta(v).$$

Now taking the v derivative of (15) we obtain (similarly to the computation in the previous section)

$$\begin{aligned} v(\theta(v)) = \mathrm{tr}(\nabla_v \nabla v) - v(\kappa) &= \sum f^i(\nabla_{e_i}(\kappa v) - \nabla_{\nabla_{e_i} v} v + \mathrm{Riem}(e_i, v)v) - v(\kappa) \\ &= \kappa \mathrm{tr}(\nabla v) + v(\kappa) - \mathrm{tr}(\nabla v \cdot \nabla v) - \mathrm{Ric}(v,v) - v(\kappa). \end{aligned}$$

And thus we have derived the *general form of Raychaudhuri's equation* (compare Theorem 11)

$$v(\theta(v)) = \kappa \operatorname{tr}(\nabla v) - \operatorname{tr}(\nabla v \cdot \nabla v) - \operatorname{Ric}(v, v). \tag{16}$$

2.3. **Interpretation.** In an infinitesimal neighborhood of a point p, for the point $p + \delta p$, the vector field v is approximately $v_p + (\nabla v)_p \cdot \delta p$. This shows that the vector field v, or "moving along the geodesics in the congruence", moves the observers chiefly by translation by v_p. But nearby observers see a slightly different motion given by $(\nabla v)_p$. The expansion, the trace of ∇v (when v is affinely parametrized), captures the infinitesimal dilation. If $\theta(v)$ is positive, then on average nearby observers will move further away from the geodesic through p. Conversely, if $\theta(v)$ is negative, then on average nearby observers will move toward the geodesic through p.

We also remark that Raychaudhuri's equation is of course closely tied to the Jacobi equation for Jacobi fields, which capture also behavior of variations within a one-parameter family of geodesics. The details we leave the reader to consider.

3. Application to Lorentzian geometry

In this section we apply and extend the theory developed above to Lorentzian manifolds and submanifolds thereof. Some of the discussion can also be carried forth for Riemannian manifolds, but many of the discussions below require a positive definite inner product to be available on certain subspaces, and hence cannot easily be generalized to more general pseudo-Riemannian manifolds.

3.1. **Time-like congruence.** Let $\mathcal{M}$ be a manifold, equipped with a Lorentzian metric g; we let ∇ be the associated Levi-Civita connection. We shall assume that $\mathcal{C}$ is a geodesic congruence; furthermore we assume that the geodesics constituting $\mathcal{C}$ are all time-like.

Returning to §2.2, that v is non-degenerate means that its orthogonal complement is transverse to v. Therefore we can naturally select $v^\flat$ to be the one-form μ, and identify $\mathcal{H}$ with $\{v\}^\perp$. As g is Lorentzian and v is time-like, $\mathcal{H}$ is equipped with a *positive definite inner product* which we will call h (to distinguish it from g). We are led to the formula

$$B(v,\zeta) = \nabla_\zeta v - \frac{g(v,\nabla_\zeta v)}{g(v,v)} v = \nabla_\zeta v - \frac{1}{2}\nabla_\zeta(\ln|g(v,v)|)v.$$

Observe that $\nabla_v g(v,v) = 2g(v,\nabla v) = 2\kappa g(v,v)$. As discussed previously, $\zeta \mapsto B(v,\zeta)$ can be interpreted as a mapping from $\{v\}^\perp$ to itself; relative to the inner product h, we can decompose $B(v,-)$ algebraically:

$$B(v,-) = \frac{1}{n-1}\theta(v)\mathrm{Id} + \sigma(v) + \omega(v). \tag{17}$$

The first term, the expansion, is the pure-trace part. The trace-free part of $B(v,—)$ is split into the self-adjoint part σ and the anti-self-adjoint part ω.

Definition 18. We refer to the trace-free self-adjoint part $\sigma(v)$ as the *shear* of the congruence $\mathcal{C}$; and the anti-self-adjoint part $\omega(v)$ as the *twist*. Note that these are only defined in the presence of an inner product on $\mathcal{H}$.

The metric h can be written as

$$h = g - \frac{1}{g(v,v)} v^\flat \otimes v^\flat \tag{19}$$

Denote by $\check{\omega}(v)$ the bilinear form $\check{\omega}(v)(\xi,\zeta) = h(\omega(v)(\xi),\zeta)$, and similarly $\check{\sigma}(v)$ the bilinear form for $\sigma(v)$, a direct computation finds

$$\check{\omega}(v) = \frac{1}{2} d(v^\flat) - \frac{1}{4} d(\ln|g(v,v)|) \wedge v^\flat.$$

By construction $\iota_v \check{\omega}(v) = 0$, and hence by Frobenius' theorem

Proposition 20. *The twist $\omega(v)$ vanishes if and only if the distribution $\mathcal{H} = \{v\}^\perp$ is locally integrable; in other words, $\omega(v)$ vanishes if and only if v is locally hypersurface orthogonal.*

We can compute also the evolution equation for $\check{\omega}(v)$. The computation is easiest in index notation:

$$\begin{aligned}\nabla_v(\nabla_a v_b - \nabla_b v_a) &= v^c(\nabla_a\nabla_c v_b + R_{acdb}v^d - \nabla_b\nabla_c v_a - R_{bcda}v^d) \\ &= \nabla_a(\kappa v_b) - \nabla_a v^c \nabla_c v_b - \nabla_b(\kappa v_a) + \nabla_b v^c \nabla_c v_a \\ &= \kappa(dv^\flat)_{ab} + (d\kappa \wedge v^\flat)_{ab} + \nabla_b v^c \nabla_c v_a - \nabla_a v^c \nabla_c v_b;\end{aligned}$$

and

$$\begin{aligned}\nabla_v(d(\ln|g(v,v)|) \wedge v^\flat)_{ab} &= (d(\ln|g(v,v)|) \wedge \kappa v^\flat)_{ab} + (d\nabla_v(\ln|g(v,v)|) \wedge v^\flat)_{ab} \\ &\quad - \nabla_a v^c \nabla_c(\ln|g(v,v)|) v_b - \nabla_b v^c \nabla_c(\ln|g(v,v)|) v_a \\ &= \kappa(d(\ln|g(v,v)|) \wedge v^\flat)_{ab} + 2(d\kappa \wedge v^\flat)_{ab} \\ &\quad - \nabla_a v^c \nabla_c(\ln|g(v,v)|) v_b - \nabla_b v^c \nabla_c(\ln|g(v,v)|) v_a.\end{aligned}$$

Putting these together and using the decomposition (17)

$$\nabla_v \check{\omega}(v) = \kappa\check{\omega}(v) - \frac{2}{n-1}\theta\check{\omega}(v) - \sigma\cdot\check{\omega}(v) - \omega\cdot\check{\sigma}(v). \tag{21}$$

A consequence of this is that if the twist $\omega(v)$ vanishes at some point p, then it vanishes along the geodesic through p.

Proposition 22. *Suppose $(\mathcal{M}, g)$ is a Lorentzian manifold equipped with a time-like geodesic congruence $\mathcal{C}$. Assume further that*

- *there exists a section of $\mathcal{C}$ that is orthogonal to $\mathcal{V}$ everywhere;*
- $\mathrm{Ric}(v,v) \geq 0$ *for every v generating $\mathcal{C}$.*

Then letting v be an affinely parametrized geodesic vector field generating $\mathcal{V}$, we have that $\theta(v)$ is monotonically decreasing in the direction of v.

Proof. By Theorem 11, we have

$$v(\theta(v)) = -\mathrm{tr}(\nabla v \cdot \nabla v) - \mathrm{Ric}(v,v).$$

Our assumption on hypersurface orthogonality initially, coupled with (21), implies that the twist vanishes everywhere, and hence ∇v is everywhere self-adjoint. Hence $\nabla v \cdot \nabla v$ is positive semi-definite. This shows $v(\theta(v)) \leq 0$. □

3.2. **Null congruences.** For our next application, we will look at *null hypersurfaces* in Lorentzian manifolds. More precisely, let $\bar{\mathcal{M}}$ be an $(n+1)$-dimensional manifold, equipped with a Lorentzian metric g. We shall let $\mathcal{M}$ be a *null hypersurface*. A convenient definition is this:

Definition 23. $\mathcal{M} \subseteq \bar{\mathcal{M}}$ is a *null hypersurface* if at every point $p \in \mathcal{M}$, there exists an open neighborhood $N \ni p$ in $\bar{\mathcal{M}}$, and a defining function $u : N \to \mathbb{R}$ such that

- $du \neq 0$ on N,
- $\mathcal{M} \cap N = u^{-1}(\{0\})$,
- $g^{-1}(du, du) = 0$ on $\mathcal{M} \cap N$.

Note that by definition, the pull-back of the full space-time metric g onto $\mathcal{M}$ is no longer a pseudo-Riemannian metric: it is degenerate. (This is a second, equivalent definition of a null hypersurface: a codimension 1 submanifold for which the pull-back of the space-time metric is degenerate.) An important consequence is that

Proposition 24. *Null hypersurfaces are ruled by null geodesics.*

Proof. It suffices to work locally. Let u be a local defining function of the null hypersurface $\mathcal{M}$. Consider the vector field $v = (du)^\sharp$. Our hypothesis implies that $g(v,v) = 0$ and $v \neq 0$. By definition along $\mathcal{M}$ we have $v(u) = du(v) = g^{-1}(du,du) = 0$, so v is tangent to the level sets of u, and hence tangent to $\mathcal{M}$.

We next show that v is geodesic along $\mathcal{M}$. To do so, we note that $\nabla_v v = \nabla_v (du)^\sharp$, and, using that the metric Hessian of a scalar function is symmetric,

$$(\nabla_v du)_a = \nabla^b u \nabla_b \nabla_a u = \nabla^b u \nabla_a \nabla_b u = \frac{1}{2}\nabla_a(\nabla^b u \nabla_b u).$$

Since $g^{-1}(du,du) = 0$ along $\mathcal{M}$, and v is tangent to $\mathcal{M}$, we see that $g(\nabla_v v, v) = \frac{1}{2} v(g^{-1}(du,du)) = 0$. In particular, this means that $(\nabla_v v)(u) = 0$ along $\mathcal{M}$, and hence $\nabla_v v$ is tangent to $\mathcal{M}$. Furthermore, we see that given *any* tangent vector w to $\mathcal{M}$, that $g(\nabla_v v, w) = 0$ also. This forces $\nabla_v v \propto v$ along $\mathcal{M}$, or that v is geodesic. □

Proposition 25. *Let v be a null geodesic vector field along $\mathcal{M}$. Then for any X tangent to $\mathcal{M}$, we have $\nabla_X v$ is also tangent to $\mathcal{M}$. As a consequence, $\nabla_v X$ is also tangent to $\mathcal{M}$.*

Proof. The first claim follows from the fact that $g(\nabla_X v, v) = \frac{1}{2}\nabla_X g(v,v) = 0$ since $g(v,v) = 0$.

For the second claim, observe that if X and v are both tangent to $\mathcal{M}$, their commutator $[X,v]$ is also tangent to $\mathcal{M}$. As the Levi-Civita connection is torsion free, we can write $\nabla_v X = \nabla_X v + [v,X]$ and hence $\nabla_v X$ is also tangent to $\mathcal{M}$. □

Now, because g pulls back to a degenerate bilinear form on $\mathcal{M}$, there is no natural "induced Levi-Civita connection" on $\mathcal{M}$. Instead, for each field of transverse vectors k along $\mathcal{M}$, there is an associated induced connection D.

Definition 26. Let k be a vector field defined on $\mathcal{M}$ that is transverse to $\mathcal{M}$, and v a null geodesic vector field on $\mathcal{M}$, together they induce a linear connection D on $\mathcal{M}$ given by

$$D_X Y = \nabla_X Y - \frac{g(\nabla_X Y, v)}{g(k,v)} k.$$

It is clear that the definition is independent of the choice of v (since $\mathcal{M}$ only has one tangent null direction). Because of Proposition 25, we see that *for any null vector field v and any X tangent to $\mathcal{M}$, the values of $D_X v$ and $D_v X$ are well-defined independently of the choice of k.* With an arbitrary choice of k, we see that $\mathcal{M}$ is an n-dimensional manifold with an induced linear connection D for which v is geodesic, and therefore our previous discussion for geodesic congruences can be applied.

Additionally, as $D_X v$ is independent of the choice of k and is equal to $\nabla_X v$, we have that the tensor field B (a section of $\mathcal{V}^* \otimes \mathcal{H}^* \otimes \mathcal{H}$) is well-defined independent of the choice of k. We may therefore extend Theorem 8 to read:

Theorem 27. *Let $\bar{\mathcal{M}}$ be a Lorentzian manifold and $\mathcal{M}$ a null hypersurface. Let $\mathcal{V}$ be the one-dimensional distribution of null vectors in $\mathsf{T}\mathcal{M}$, and set $\mathcal{H} = \mathsf{T}\mathcal{M}/\mathcal{V}$. Then there exists a section B of $\mathcal{V}^* \otimes \mathcal{H}^* \otimes \mathcal{H}$ such that given an equivalence class $[\xi] \in \mathcal{H}$ with representative $\xi \in \mathsf{T}\mathcal{M}$, and v a section of $\mathcal{V}$, the vector $\nabla_\xi v$ belongs to the equivalent class $B(v, [\xi])$.*

In this context, we refer to the tensor field B as the null second fundamental form *of the hypersurface $\mathcal{M}$. The corresponding section θ of $\mathcal{V}^*$ is called the* null expansion *of $\mathcal{M}$.*

In contrast to the case with time-like congruences of Lorentzian manifolds, the fact that the pull back of g to $\mathcal{M}$ is degenerate, with kernel $\mathcal{V}$, means that *g induces a positive definite inner product on $\mathcal{H}$*. More precisely, observe that given $\xi, \xi' \in \mathsf{T}_p\mathcal{M}$ and v the null geodesic field, we have that for any $\lambda, \lambda' \in \mathbb{R}$ the scalar product $g(\xi + \lambda v, \xi' + \lambda' v)$ is independent of λ, λ'. This implies that g factors through the equivalence classes $\mathcal{H}$. Relative to this inner product, we can also factor

$$B(v,-) = \frac{1}{n-1}\theta(v)\mathrm{Id}_{\mathcal{H}} + \sigma(v) + \omega(v) \tag{28}$$

where $\sigma(v)$ (null shear) is the trace-free self-adjoint part of B, and $\omega(v)$ (null twist) is the anti-self-adjoint part.

To better analyze this decomposition, it is easier to get a more convenient representation of $B(v,x)$. Given $[x],[y]$ in $\mathcal{H}$, we want to think about the bilinear form given by the mapping

$$\check{B} : ([x],[y]) \mapsto \langle B(v,[x]),[y]\rangle \tag{29}$$

where the inner product is the one induced by g. But as discussed above, this product can be set to equal to $g(Z,Y)$ for any Y in the equivalence class of $[y]$ and Z in the equivalence class of $B(v,[x])$, and by Theorem 27 if we take any representatives (X,Y) of $([x],[y])$ the value is exactly equal to $g(\nabla_X v, Y)$. We are therefore led to study the bilinear form $(\mathsf{T}\mathcal{M})^2 \ni (X,Y) \mapsto g(\nabla_X v, Y)$. Then $\sigma(v)$ corresponds to the trace-free symmetric part of this bilinear form, and $\omega(v)$ corresponds to the anti-symmetric part.

We wish to consider the evolution of this bilinear form. For this we need to take derivatives in the v direction. By Proposition 25, we find that for any X tangent to $\mathcal{M}$,

$$D_v(X + \lambda v) = D_v X + v(\lambda)v + \lambda D_v v.$$

And hence two vector fields in the same equivalence class of $\mathcal{H}$ have v derivatives in the same equivalence class. This implies that D_v is well-defined on sections of the bundle $\mathcal{H}$. Furthermore, considering the bilinear form $\check{B}$ defined in (29), its covariant derivative can be computed to be

$$\begin{aligned}(D_v\check{B})([x],[y]) &= v(\check{B}([x],[y])) - \check{B}(D_v[x],[y]) - \check{B}([x],D_v[y]) \\ &= v(g(\nabla_x v,y)) - \check{B}([D_v x],[y]) - \check{B}([x],[D_v y]) \\ &= v(g(\nabla_x v,y)) - g(\nabla_{\nabla_v x} v,y) - g(\nabla_x v,\nabla_v y).\end{aligned}$$

We can rewrite this as

$$\begin{aligned}(D_v\check{B})([x],[y]) &= g(\mathrm{Riem}(x,v)v,y) + g(\nabla_x\nabla_v v,y) - g(\nabla_{\nabla_x v} v,y) \\ &= g(\mathrm{Riem}(x,v)v,y) + \kappa g(\nabla_x v,y) - g(\nabla_{\nabla_x v} v,y).\end{aligned}$$

The symmetry properties of the Riemann curvature tensor means that if x,x' and y,y' are two representatives of $[x]$ and $[y]$ respectively, then

$$g(\mathrm{Riem}(x,v)v,y) = g(\mathrm{Riem}(x',v)v,y');$$

furthermore we know that $g(\mathrm{Riem}(x,v)v,y) = g(\mathrm{Riem}(y,v)v,x)$, and hence there exists a *symmetric bilinear form* $\mathfrak{R}$ on $\mathcal{H}$ satisfying

$$\mathfrak{R}([x],[y]) = g(\mathrm{Riem}(x,v)v,y). \tag{30}$$

And finally we may write

$$D_v\check{B} = \mathfrak{R} + \kappa\check{B} - \check{B}(B(v,-),-). \tag{31}$$

If we decompose

$$\check{B} = \frac{1}{n-1}\theta(v)g + \check{\sigma} + \check{\omega}$$

then we find that we can expand

$$\begin{aligned}\check{B}(B(v,-),-) = \frac{1}{(n-1)^2}\theta(v)^2 g + \frac{2}{n-1}\theta(v)\check{\sigma} + \frac{2}{n-1}\theta(v)\check{\omega} \\ + g(\sigma(-),\sigma(-)) - g(\omega(-),\omega(-)) + g(\omega(-),\sigma(-)) - g(\sigma(-),\omega(-)).\end{aligned}$$

Before continuing further, however, observe that in this setting the null twist ω has a meaning similar to how the twist in the time-like setting measures hypersurface orthogonality. As discussed before, $\check{B}(X,Y) = g(\nabla_X v,y)$ given a representative v of $\mathcal{V}$. So

$$\check{\omega} = d(v^\flat)|_{\mathcal{M}}.$$

Letting u be again the defining function of $\mathcal{M}$, then we find we can always write $v^\flat = f\ du$ for some scalar function f. So we have

$$\check{\omega} = df \wedge du|_{\mathcal{M}} = 0$$

always. So we can conclude that in our setting where we consider a null geodesic congruence *attached to a null hypersurface*, the null twist automatically vanishes. (In more general settings where a family of not-necessarily integrable null geodesic congruence is studied, the twist will play a role.) Armed with this knowledge we find

$$\check{B} = \frac{1}{n-1}\theta(v)g + \check{\sigma} \tag{32}$$

and (31) becomes

$$D_v\check{B} = \mathfrak{R} + \kappa\check{B} - \frac{1}{(n-1)^2}\theta(v)^2 g - \frac{2}{n-1}\theta(v)\check{\sigma} - g(\sigma(-),\sigma(-)). \tag{33}$$

We can split this into the evolution for $\theta(v)$ and the evolution for $\check{\sigma}$. To do so, we need to compute the derivative $D_v g$, where g is interpreted as the induced bilinear form on $\mathrm{T}\mathcal{M}$ from the Lorentzian metric on $\bar{\mathcal{M}}$. By definition, we have

$$(D_v g)(X,Y) = v(g(X,Y)) - g(D_v X, Y) - g(X, D_v Y).$$

But as we have discussed before, the vector $D_v X$ is equal to $\nabla_v X$, the latter computed using the ambient Levi-Civita connection, thanks to Proposition 25. Therefore the metric compatibility of the Levi-Civita connection actually produces the statement that

$$D_v g = 0. \tag{34}$$

Next, the general Raychaudhuri's equation (16) can be re-written in the following form (using that $\omega = 0$)

$$v\Big(\theta(v)\Big) = \kappa\theta(v) - \frac{1}{n-1}\theta(v)^2 - \mathrm{tr}(\sigma\circ\sigma) - \mathrm{Ric}(v,v). \tag{35}$$

Plugging this into (33), we find the following equation:

$$\begin{aligned} D_v\check{\sigma} = \kappa\check{\sigma} - \frac{2}{n-1}\theta(v)\check{\sigma} + \mathfrak{R} + \frac{1}{n-1}\mathrm{Ric}(v,v)g \\ - g(\sigma\circ\sigma(-),-) + \frac{1}{n-1}\mathrm{tr}(\sigma\circ\sigma)g. \end{aligned} \tag{36}$$

Remark 37. When $n = 3$, this last equation simplifies. Observe that if V is a two-dimensional inner product space, and σ a self-adjoint, trace-free operator, and ω and anti-self-adjoint operator, we have that relative to a standard basis

$$\sigma = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}, \qquad \omega = \begin{pmatrix} 0 & c \\ -c & 0 \end{pmatrix}$$

and hence

$$\sigma^2 = \begin{pmatrix} a^2+b^2 & \\ & a^2+b^2 \end{pmatrix}, \qquad \omega = \begin{pmatrix} -c^2 & \\ & -c^2 \end{pmatrix}$$

are automatically pure trace! So in the physical space-time (where $n+1 = 4$), we have that the last two factors cancel each other, as the trace-free part of $\sigma\circ\sigma$ vanishes. In higher dimensions however these two terms may be present non-trivially.

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