The goal of this short note is to record elementary proofs of interpolation inequalities of the form

\[ \|f\|_q \lesssim \|f\|_p^\theta \cdot |f|_0,\beta^{1-\theta} \]

where \( f \) is a real valued smooth function defined on some bounded interval, the norms \( \| - \|_p \) are the \( L^p \) norms, and the seminorm \( | - |_{0,\beta} \) is the Hölder \( C^{0,\beta} \) seminorm. We shall take \( 1 \leq p < q \leq \infty \) and \( \beta \in (0,1] \). The parameter \( \theta \) takes values in \((0,1)\). Similarly inequalities are presumably available on domains in \( \mathbb{R}^n \), but for simplicity of argument and simplicity of ideas we will only consider the one dimensional case here.

The inequality (1) should be compared with the Gagliardo-Nirenberg-Sobolev-Morrey inequalities, which provide estimates of the form

\[ |f|_{0,\beta} \lesssim \|f\|_p^\theta \|f^{(k)}\|_q^{1-\theta} \]

here \( f^{(k)} \) is the \( k \)th derivative of \( f \), with \( k \geq 1 \), and \( p,q \in [1,\infty] \). They guiding principle behind all such inequalities is the idea of a Sobolev scale. Given \( \alpha,\beta \in [0,\infty) \) and \( p,q \in [1,\infty] \), we write

\[ (\alpha,p) \preceq (\beta,q) \iff \alpha - \frac{1}{p} \leq \beta - \frac{1}{q}. \]

One can check that this defines a total preorder on \([0,\infty) \times [1,\infty] \); this ordering is the Sobolev scale. We can associate to the Sobolev seminorm \( f \mapsto \|f^{(k)}\|_q \) the scale \((k,q)\), and to the Hölder seminorm \( f \mapsto |f^{(k)}|_{0,\beta} \) the scale \((k+\beta,\infty)\). The basic heuristic behind interpolation inequalities on the Sobolev scale is the idea that given

\[ (\alpha,p) \preceq (\beta,q) \preceq (\gamma,r) \]

(and some other technical assumptions) a seminorm at the scale \((\beta,q)\) can be interpolated between the seminorms of the scales \((\alpha,p)\) and \((\gamma,r)\).

1. Hölder seminorms

Given \( f : I \to \mathbb{R} \), its Hölder seminorm is usually defined as

\[ |f|_{0,\beta} := \sup_{x,y \in I} \frac{|f(x) - f(y)|}{|x-y|^{\beta}}. \]

The function \( f \) is said to be of class \( C^{0,\beta} \) if \( |f|_{0,\beta} < \infty \).

The following is a convenient equivalent definition:
Definition 4. A continuous function is of class \( C^{0,\beta} \) if there exists \( M > 0 \) such that for any interval \( J \subseteq I \) and every \( x \in J \), we have
\[
|f(x) - \frac{1}{|J|} \int_J f| \leq M|J|^\beta.
\]

Proof. Given \( f \) is continuous, it attains its average value on \( J \), call this point \( x_0 \). Then \( |f(x) - f(x_0)| \leq |f|_{0,\beta}|x - x_0|^\beta \). This shows that “original definition” implies “new definition”.

Suppose now \( f \) satisfies “new definition”, then given \( x < y \), let \( J = [x, y] \) and estimate \( |f(x) - f(y)| \leq |f(x) - \frac{1}{|J|} \int_J f| + |f(y) - \frac{1}{|J|} \int_J f| \) and the “original definition” follows. \( \square \)

Now, supposing \( f \) is class \( C^{0,\beta} \), then it is continuous, and hence for any interval \( J \) we have \( f(J) \) is an interval. This means that
\[
\sup_x |f(x) - \frac{1}{|J|} \int_J f| \geq \frac{1}{2}|f(J)|.
\]
and hence up to universal constant, we have that for every interval \( J \) that
\[
(5) \quad |f(J)| \leq |J|^{1/\beta} |f|_{0,\beta}.
\]

2. Interpolation

Now let \( h : I \to \mathbb{R} \) be a \( \mathcal{C}^{0,\beta} \) function. Suppose that \( h(x_0) = 0 \) for some \( x_0 \in I \). Let \( x_n \) be a sequence maximizing \( |h| \), by Bolzano-Weierstrass we can assume \( x_n \) converge. Consider the set \( \{x : \frac{1}{2} \sup \{|h| \leq h(x)\}\} \). Since \( x_n \) converges, it is eventually in one of its connected components, which we call \( J \). Observe that \( \sup_{x \in J} |h(x)| = \|h\|_\infty \). As \( I \) is connected, by the intermediate value theorem (against the point \( x_0 \)), we must have \( \inf_{x \in J} |h(x)| = \frac{1}{2}\|h\|_\infty \). This shows that \( |h(J)| = \frac{1}{2}\|h\|_\infty \).

We can now apply (5) to conclude
\[
(6) \quad \frac{1}{2}\|h\|_\infty \leq |h(J)| \leq |J|^{1/\beta} |h|_{0,\beta}.
\]

This we rewrite as
\[
\left(\frac{1}{2}\|h\|_\infty\right)^{p+1/\beta} \leq |h|_{0,\beta}^{1/\beta} \left(\frac{1}{2}\|h\|_\infty\right)^p |J|.
\]
Note that as \( |h| \) is at least \( \frac{1}{2}\|h\|_\infty \) on \( J \), the product on the right is a lower bound for the \( \int_J |h|^p \leq \int_J |h|^{p_\beta} \). And so we conclude
\[
(7) \quad \left(\frac{1}{2}\|h\|_\infty\right)^{p+1/\beta} \leq |h|_{0,\beta}^{1/\beta} |h|_{p_\beta}^p.
\]

This shows:

Theorem 8. There exists a universal constant \( C \) such that, whenever \( I \) is a bounded interval, and \( h : I \to \mathbb{R} \) of class \( C^{0,\beta} \) with \( 0 \in h(I) \), we have
\[
\|h\|_\infty \leq C|h|_{0,\beta}^{1/\beta} |h|_{p_\beta}^{p_\beta/\beta+1}.
\]
Note that the constant is independent of \( I \).
The $\|h\|_q$ version follows immediately after applying the interpolation inequality
\[ \int_I |h|^q \leq \|h\|_\infty^{q-p} \int_I |h|^p. \]

3. Dropping the regularity

A next question is, can we get a similar expression interpolating with lower derivatives? That is, given $f \in C_0^\infty(\mathbb{R})$, can we estimate
\[ \|f'\|_\infty \lesssim \|f\|_{\theta} |f'|_{1-\theta}^0? \]
It turns out the answer is yes. Going back to (6), which we apply to $h = f'$, we find
\[ \|f'\|_\infty^{1+1/\beta} \leq \|f\|_\infty |f'|_{0,\beta}^{1/\beta}. \]
The key point is that $|f'|$ is bounded below by $\frac{1}{2} |f'|_\infty$ on $|J|$, and hence $f'$ is signed. This means we can estimate
\[ \|f'\|_\infty |J| \leq 2 \int_J f' \leq 4 \|f\|_\infty. \]
And so we have the estimate
\[ \|f'\|_\infty^{1+1/\beta} \leq \|f\|_\infty |f'|_{0,\beta}^{1/\beta}. \]
This can now be further upgraded, using the $\beta = 1$ version of what we proved before
\[ \|f'\|_\infty^{1+1/\beta} \leq \|f\|_p^{\frac{1}{p+1}} \|f'\|_0^{\frac{1}{p+1}} \|f|_{0,\beta}^{1/\beta}. \]
Cancelling we find
\[ \|f'\|_\infty \leq \|f\|_p \|f'|_{0,\beta}^{1/\beta}. \]
Applying to the higher derivatives we find also
\[ \|f^{(k)}\|_\infty \leq \|f^{(k-1)}\|_p \|f^{(k)}\|_{0,\beta}^{\frac{1}{p+1}} . \]
We next combine this with the Gagliardo-Nirenberg-Sobolev interpolation, one version of which states
\[ \|f^{(k)}\|_\infty \leq \|f^{(k-1)}\|_p \|f\|_0^{\frac{k}{p+1}} \|f^{(k)}\|_{0,\beta}^{\frac{1}{p+1}} . \]
This yields after some computation
\[ \|f^{(k)}\|_\infty \leq \|f\|_p^{\frac{k}{p+1}} |f^{(k)}|_{0,\beta}^{\frac{1}{p+1}} . \]
Inserting this into the Gagliardo-Nirenberg-Sobolev inequality
\[ \|f\|_\infty \leq \|f\|_p^{\frac{k}{p+1}} |f^{(k)}|_{\infty}^{\frac{1}{p+1}} . \]
we conclude, at the end of the day, the following estimate:
\[ \|f\|_\infty \leq \|f\|_p^{\frac{k}{p+1}} |f^{(k)}|_{0,\beta}^{\frac{1}{p+1}} . \]