SOME INTERPOLATION INEQUALITIES

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The goal of this short note is to record elementary proofs of interpolation inequalities of the form

$$\|f\|_q \lesssim \|f\|_p^\theta \cdot |f|_{0,\beta}^{1-\theta} \tag{1}$$

where f is a real valued smooth function defined on some bounded interval, the norms $\|{-}\|_p$ are the L^p norms, and the seminorm $|{-}|_{0,\beta}$ is the Hölder $C^{0,\beta}$ seminorm. We shall take $1 \leq p < q \leq \infty$ and $\beta \in (0,1]$. The parameter θ takes values in $(0,1)$. Similarly inequalities are presumably available on domains in $\mathbb{R}^n$, but for simplicity of argument and simplicity of ideas we will only consider the one dimensional case here.

The inequality (1) should be compared with the Gagliardo-Nirenberg-Sobolev-Morrey inequalities, which provide estimates of the form

$$|f|_{0,\beta} \lesssim \|f\|_p^\theta \|f^{(k)}\|_q^{1-\theta} \tag{2}$$

here $f^{(k)}$ is the kth derivative of f, with $k \geq 1$, and $p,q \in [1,\infty]$. They guiding principle behind all such inequalities is the idea of a *Sobolev scale*. Given $\alpha, \beta \in [0,\infty)$ and $p,q \in [1,\infty]$, we write

$$(\alpha,p) \leqq (\beta,q) \iff \alpha - \frac{1}{p} \leq \beta - \frac{1}{q}. \tag{3}$$

One can check that this defines a total preorder on $[0,\infty)\times[1,\infty]$; this ordering is the Sobolev scale. We can associate to the Sobolev seminorm $f \mapsto \|f^{(k)}\|_q$ the scale (k,q), and to the Hölder seminorm $f \mapsto |f^{(k)}|_{0,\beta}$ the scale $(k+\beta,\infty)$. The basic heuristic behind interpolation inequalities on the Sobolev scale is the idea that given

$$(\alpha,p) \lneqq (\beta,q) \lneqq (\gamma,r)$$

(and some other technical assumptions) a seminorm at the scale (β,q) can be interpolated between the seminorms of the scales (α,p) and (γ,r).

1. Hölder seminorms

Given $f : I \to \mathbb{R}$, its Hölder seminorm is usually defined as

$$|f|_{0,\beta} \coloneqq \sup_{x\neq y\in I} \frac{|f(x)-f(y)|}{|x-y|^\beta}.$$

The function f is said to be of class $C^{0,\beta}$ if $|f|_{0,\beta} < \infty$.

The following is a convenient equivalent definition:

Definition 4. A continuous function is of class $C^{0,\beta}$ if there exists $M > 0$ such that for any interval $J \subseteq I$ and every $x \in J$, we have

$$\Big|f(x) - \frac{1}{|J|}\int_J f\Big| \leq M|J|^\beta.$$

Proof. Given f is continuous, it attains its average value on J, call this point x_0. Then $|f(x) - f(x_0)| \leq |f|_{0,\beta}|x - x_0|^\beta$. This shows that "original definition" implies "new definition".

Suppose now f satisfies "new definition", then given $x < y$, let $J = [x,y]$ and estimate $|f(x) - f(y)| \leq |f(x) - \frac{1}{|J|}\int_J f| + |f(y) - \frac{1}{|J|}\int_J f|$ and the "original definition" follows. □

Now, supposing f is class $C^{0,\beta}$, then it is continuous, and hence for any interval J we have $f(J)$ is an interval. This means that

$$\sup_x \in J\Big|f(x) - \frac{1}{|J|}\int_J f\Big| \geq \frac{1}{2}|f(J)|.$$

and hence up to universal constant, we have that for every interval J that

$$|f(J)| \lesssim |J|^\beta |f|_{0,\beta}. \tag{5}$$

2. Interpolation

Now let $h : I \to \mathbb{R}$ be a $C^{0,\beta}$ function. Suppose that $h(x_0) = 0$ for some $x_0 \in I$. Let x_n be a sequence maximizing $|h|$, by Bolzano-Weierstrass we can assume x_n converge. Consider the set $\{x : \frac{1}{2}\sup|h| \leq h(x)\}$. Since x_n converges, it is eventually in one of its connected components, which we call J. Observe that $\sup_{x\in J}|h(x)| = \|h\|_\infty$. As I is connected, by the intermediate value theorem (against the point x_0), we must have $\inf_{x\in J}|h(x)| = \frac{1}{2}\|h\|_\infty$. This shows that $|h(J)| = \frac{1}{2}\|h\|_\infty$.

We can now apply (5) to conclude

$$\frac{1}{2}\|h\|_\infty \leq |h(J)| \lesssim |J|^\beta |h|_{0,\beta}. \tag{6}$$

This we rewrite as

$$(\frac{1}{2}\|h\|_\infty)^{p+1/\beta} \lesssim |h|_{0,\beta}^{1/\beta}(\frac{1}{2}\|h\|_\infty)^p|J|.$$

Note that as $|h|$ is at least $\frac{1}{2}\|h\|_\infty$ on J, the product on the right is a lower bound for the $\int_J |h|^p \leq \int_I |h|^p$. And so we conclude

$$(\frac{1}{2}\|h\|_\infty)^{p+1/\beta} \lesssim |h|_{0,\beta}^{1/\beta}\|h\|_p^p. \tag{7}$$

This shows:

Theorem 8. *There exists a universal constant C such that, whenever I is a bounded interval, and $h : I \to \mathbb{R}$ of class $C^{0,\beta}$ with $0 \in h(I)$, we have*

$$\|h\|_\infty \leq C|h|_{0,\beta}^{\frac{1}{p\beta+1}}\|h\|_p^{\frac{p\beta}{p\beta+1}}.$$

Note that the constant is independent of I.

The $\|h\|_q$ version follows immediately after applying the interpolation inequality

$$\int_I |h|^q \leq \|h\|_\infty^{q-p} \int_I |h|^p.$$

3. Dropping the regularity

A next question is, can we get a similar expression interpolating with lower derivatives? That is, given $f \in C_0^\infty(\mathbb{R})$, can we estimate

$$\|f'\|_\infty \lesssim \|f\|_p^\theta |f'|_{0,\beta}^{1-\theta}?$$

It turns out the answer is yes. Going back to (6), which we apply to $h = f'$, we find

$$\|f'\|_\infty^{1+1/\beta} \lesssim \|f'\|_\infty |J| |f'|_{0,\beta}^{1/\beta}.$$

The key point is that $|f'|$ is bounded below by $\frac{1}{2}\|f'\|_\infty$ on $|J|$, and hence f' is *signed*. This means we can estimate

$$\|f'\|_\infty |J| \leq 2\left|\int_J f'\right| \leq 4\|f\|_\infty.$$

And so we have the estimate

$$\|f'\|_\infty^{1+1/\beta} \lesssim \|f\|_\infty |f'|_{0,\beta}^{1/\beta}. \tag{9}$$

This can now be further upgraded, using the $\beta = 1$ version of what we proved before

$$\|f'\|_\infty^{1+1/\beta} \lesssim |f|_{0,1}^{\frac{1}{p+1}} \|f\|_p^{\frac{p}{p+1}} |f'|_{0,\beta}^{1/\beta} \lesssim \|f'\|_\infty^{1/(p+1)} \|f\|_p^{p/(p+1)} |f'|_{0,\beta}^{1/\beta}$$

Cancelling we find

$$\|f'\|_\infty \lesssim \|f\|_p^{\frac{\beta p}{\beta p+p+1}} |f'|_{0,\beta}^{\frac{p+1}{\beta p+p+1}}. \tag{10}$$

Applying to the higher derivatives we find also

$$\|f^{(k)}\|_\infty \lesssim \|f^{(k-1)}\|_p^{\frac{\beta p}{\beta p+p+1}} |f^{(k)}|_{0,\beta}^{\frac{p+1}{\beta p+p+1}}. \tag{11}$$

We next combine this with the Gagliardo-Nirenberg-Sobolev interpolation, one version of which states

$$\|f^{(k-1)}\|_p \lesssim \|f^{(k)}\|_\infty^{\frac{p(k-1)}{pk+1}} \|f\|_p^{\frac{p+1}{pk+1}}. \tag{12}$$

This yields after some computation

$$\|f^{(k)}\|_\infty^{\beta p+pk+1} \lesssim \|f\|_p^{\beta p} |f^{(k)}|_{0,\beta}^{pk+1}. \tag{13}$$

Inserting this into the Gagliardo-Nirenberg-Sobolev inequality

$$\|f\|_\infty \lesssim \|f\|_p^{\frac{kp}{kp+1}} \|f^{(k)}\|_\infty^{\frac{1}{kp+1}} \tag{14}$$

we conclude, at the end of the day, the following estimate:

$$\|f\|_\infty \lesssim \|f\|_p^{\frac{(k+\beta)p}{(k+\beta)p+1}} |f^{(k)}|_{0,\beta}^{\frac{1}{(k+\beta)p+1}}. \tag{15}$$

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