EPIGRAPHICAL LIMITS AND FRIENDS

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Abstract. A short note rewriting some of the ideas concerning the so-called *epigraphical limits*, which turns out to be useful in convex and variational analysis.

1. Introduction

For more details on the material discussed in here, the reader should consult

> R.T. Rockafellar and R.J-B. Wets, *Variational Analysis*, Springer-Verlag (1998), https://doi.org/10.1007/978-3-642-02431-3.

We can start with some motivational examples related to variational analysis. One of the purposes of variational analysis is to understand those inputs that correspond to global minima of ($\mathbb{R}$-valued) functions. An example of the questions considered are of the following form:

Example 1.1. Let $K \subseteq \mathbb{R}^d$ and $f_\nu \to \mathbb{R}$ a net of continuous $S \to \mathbb{R}$ functions. By the extremal value theorem, each f_ν attains its minimum on K, and we may choose one such representative x_ν for each f_ν. As K is compact, x_ν has accumulation points. *Given that f_ν converges to some function f_∞ (in some yet unspecified sense), are the accumulation points of x_ν related to the minima of f_∞?*

That the answer depends on the mode of convergence of f_ν can be seen through examples. In the positive direction, we may suppose that f_ν converges to f_∞ *uniformly*. In this case, if ξ is an accumulation point of x_ν, then ξ is a minimum of f_∞.

Proof. By taking a subnet, we can assume that $x_\nu \to \xi$. Write $f_\nu(x_\nu) - f_\infty(\xi) = f_\nu(x_\nu) - f_\infty(x_\nu) + f_\infty(x_\nu) - f_\infty(\xi)$. The first difference converges to zero by uniform convergence, and the second difference converges to zero due to the continuity of f_∞ (as the uniform limit of continuous functions). And hence $f_\nu(x_\nu) \to f_\infty(\xi)$. It suffices to show $f_\infty(\xi) = \min f_\infty$.

Suppose not, then there exists $\zeta \in K$ such that $f_\infty(\xi) - f_\infty(\zeta) =: m > 0$. By uniform convergence we have that $f_\nu(\zeta)$ is eventually less than $f_\infty(\xi) - m/2$. But $f_\nu(x_\nu)$ is eventually larger than $f_\infty(\xi) - m/2$, showing that eventually x_ν is not a minimum of f_ν, a contradiction to its definition. □

We remark that only the first half of the proof really requires uniform convergence and continuity. Indeed, we have the following proposition:

Proposition 1.2. *Let $f_\nu : S \to \mathbb{R}$ a net of functions defined on a set S, and suppose $f_\nu \to f_\infty$* pointwise. *Then* $\inf f_\infty \geq \limsup(\inf f_\nu)$.

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Proof. For each $s \in S$, for each $\epsilon > 0$, we have that $f_\nu(s)$ is eventually less than $f_\infty(s)+\epsilon$. As $\inf f_\nu \leq f_\nu(s)$, we have that $\inf f_\nu$ is eventually less than $f_\infty(s)+\epsilon$. Thus $\limsup(\inf f_\nu) \leq f_\infty(s)+\epsilon$ for every $\epsilon > 0$, and hence $\limsup(\inf f_\nu) \leq f_\infty(s)$ for every $s \in S$. But this means $\inf f_\infty \geq \limsup(\inf f_\nu)$. □

On the other hand, pointwise convergence is not sufficient to guarantee a positive answer to our original problem. Consider the case $K = [-1,2] \subseteq \mathbb{R}$. For $n \in \mathbb{N}$, let $f_n : K \to \mathbb{R}$ be the piecewise linear continuous function satisfying

$$f_n(-1) = 0 \qquad f_n(-\tfrac{1}{2}) = 1 = f(0) \qquad f_n(\tfrac{1}{n}) = -1 \qquad f_n(\tfrac{2}{n}) = 1 = f_n(2).$$

The pointwise limit of this sequence is the piecewise linear continuous function satisfying

$$f_\infty(-1) = 0 \qquad f_\infty(-1/2) = 1 = f_\infty(2).$$

The functions f_n realize their minima at $x_n = \frac{1}{n}$ with $f_n(\frac{1}{n}) = -1$. But $\lim x_n = 0$ is not a minimum of f_∞, and $\lim f_n(x_n) = -1 < \inf f_\infty = 0$.

The notion of *epigraph convergence* is a notion of convergence for nets of functions defined on subsets of $\mathbb{R}^d$. By design it is weaker than (and implied by) uniform convergence. The purpose of these notes is to describe this notion, which is constructed to be compatible with the process of taking minima.

2. Convergence of nets of sets in a topological space

We begin by introducing a notion of convergence of nets of sets.

Definition 2.1. Let (X,τ) be a topological space, and A a directed set. Consider a net $(S_\alpha)_{\alpha \in A}$ of subsets of X. Given $x \in X$,

- we say that x *attracts* S_α if for every open set $O \ni x$, it is true that $S_\alpha \cap O$ is eventually non-empty;
- we say that x *repels* S_α if there exists an open set $O \ni x$ such that $S_\alpha \cap O$ is eventually empty.

We further define

$$\underline{\tau\text{-lim}}\, S_\alpha \stackrel{\text{def}}{=} \{x \in X \mid x \text{ attracts } S_\alpha\}; \tag{2.2}$$

$$\overline{\tau\text{-lim}}\, S_\alpha \stackrel{\text{def}}{=} \{x \in X \mid x \text{ does not repel } S_\alpha\}. \tag{2.3}$$

Recall that using the inclusion ordering, we can define the limits superior and inferior of a net of sets as follows:

$$\limsup S_\alpha = \bigcap_\alpha \bigcup_{\beta \geq \alpha} S_\beta$$

$$\liminf S_\alpha = \bigcup_\alpha \bigcap_{\beta \geq \alpha} S_\beta$$

We see that x repels S_α if and only if there is an open $O \ni x$ and α_0 such that $O \subseteq \complement S_\alpha$ for every $\alpha \geq \alpha_0$. This is equivalent to $x \in \operatorname{int}(\cap_{\alpha \geq \alpha_0} \complement S_\alpha)$ for some α_0. So applying De Morgan's Laws we find the similar statement

$$\overline{\tau\text{-lim}}\, S_\alpha = \bigcap_\alpha \operatorname{cl}\Big(\bigcup_{\beta \geq \alpha} S_\beta\Big). \tag{2.4}$$

(The author is not able to come up with a similar expression for $\underline{\tau\text{-lim}}\, S_\alpha$.) As the notion of "not attracting" and "repelling" are both based on the existence of certain open sets, this immediately implies

$$\text{Both } \overline{\tau\text{-lim}}\, S_\alpha \text{ and } \underline{\tau\text{-lim}}\, S_\alpha \text{ are closed sets.} \tag{2.5}$$

As "eventually" implies "frequently", we also see

$$\underline{\tau\text{-lim}}\, S_\alpha \subseteq \overline{\tau\text{-lim}}\, S_\alpha. \tag{2.6}$$

The notion of attractors and repellers can be best illustrated when $S_\alpha = \{x_\alpha\}$, where x_α is a net of points in X. In this case, $\overline{\tau\text{-lim}}\, S_\alpha$ is nothing more than the set of all accumulation points of x_α, and $x \in \underline{\tau\text{-lim}}\, S_\alpha$ if and only if x_α converges to x.

We note that it is possible for x to neither attract nor repel S_α. Hence in general $\underline{\tau\text{-lim}}\, S_\alpha$ and $\overline{\tau\text{-lim}}\, S_\alpha$ may differ. To give an example, consider the set $X = \{0, 1\}$ with the discrete topology, and the sequence $S_n = \{n \mod 2\}$. Then both points in the set neither attract nor repel S_n; in this case we have $\underline{\tau\text{-lim}}\, S_n = \emptyset$, while $\overline{\tau\text{-lim}}\, S_n = \{0, 1\}$.

Definition 2.7. Let (X, τ) be a topological space, a net S_α of subsets of X is said to τ-converge if $\underline{\tau\text{-lim}}\, S_\alpha = \overline{\tau\text{-lim}}\, S_\alpha$. In this case we denote by $\tau\text{-lim}\, S_\alpha$ the common limit.

Proposition 2.8 (Monotone convergence)**.** *If S_α is monotone by inclusion, then it τ-converges.*

Proof. (*Increasing case*). When S_α is increasing, then $\cup_{\beta \ge \alpha} S_\beta$ is independent of α. So $\overline{\tau\text{-lim}}\, S_\alpha = \mathrm{cl}(\cup S_\alpha)$. As S_α is increasing, if an open set O intersects some S_α, it must intersect all S_β with $\beta \ge \alpha$. So "not repelling" (every open $O \ni x$ frequently intersects S_α) implies "attracting" (eventually intersects).

(*Decreasing case*). As S_α is decreasing, we find $\cup_{\beta \ge \alpha} S_\beta = S_\alpha$, so $\overline{\tau\text{-lim}}\, S_\alpha = \cap_\alpha \mathrm{cl}(S_\alpha)$. It suffices to show that "not attracting" implies "repelling" in this case. The former asserts there exists an open set that frequently is disjoint from S_α. As S_α is decreasing, if $O \cap S_\alpha = \emptyset$ then $O \cap S_\beta = \emptyset$ for every $\beta \ge \alpha$. So frequently implies eventually. □

Proposition 2.9 (Localization)**.** *Let (X, τ) be a topological space, and $Y \subseteq X$ open. Given a net of subsets S_α of X, we have*

$$Y \cap \underline{\tau\text{-lim}}\, S_\alpha \subseteq \underline{\tau\text{-lim}}(S_\alpha \cap Y) \subseteq \mathrm{cl}(Y) \cap \underline{\tau\text{-lim}}\, S_\alpha,$$

$$Y \cap \overline{\tau\text{-lim}}\, S_\alpha \subseteq \overline{\tau\text{-lim}}(S_\alpha \cap Y) \subseteq \mathrm{cl}(Y) \cap \overline{\tau\text{-lim}}\, S_\alpha.$$

In particular, intersection all three with Y we obtain equality.

Proof. We focus on the attractors case; the non-repellers case is similar and omitted. For the first containment, let's fix $x \in Y \cap \underline{\tau\text{-lim}}\, S_\alpha$.

- Since $x \in Y$, given an open $O \ni x$, we have $O \cap Y$ is an open that contains x.
- Since $x \in \underline{\tau\text{-lim}}\, S_\alpha$, we have that $(O \cap Y) \cap S_\alpha$ is eventually non-empty. This also means that $O \cap (Y \cap S_\alpha)$ is eventually non-empty, showing $x \in \underline{\tau\text{-lim}}(S_\alpha \cap Y)$.

For the second containment, if $x \in \underline{\tau\text{-lim}}(S_\alpha \cap Y)$, then for every open set $O \ni x$ we have that $O \cap S_\alpha \cap Y$ is eventually non-empty. This means that $O \cap S_\alpha$ is eventually non-empty (implying $x \in \underline{\tau\text{-lim}}\, S_\alpha$) and that $O \cap Y$ is non-empty (implying $x \in \mathrm{cl}(Y)$). □

3. Epigraphical Convergence

Let (X,τ) be a topological space. Throughout $\overline{\mathbb{R}} = [-\infty,\infty]$ will denote the extended real line, equipped with the standard order topology. We will endow $X \times \overline{\mathbb{R}}$ with the product topology.

Definition 3.1 (Closed epigraphs)**.** Given $f : X \to \overline{\mathbb{R}}$ a function, its *closed epigraph* is the subset of $X \times \overline{\mathbb{R}}$ given by

$$\operatorname{epi}(f) \stackrel{\text{def}}{=} \operatorname{cl}\Big(\{(x,z) \in X \times \overline{\mathbb{R}} \mid z \geq f(x)\}\Big).$$

Similarly, a subset $F \subseteq X \times \overline{\mathbb{R}}$ is said to be a *closed epigraph* if there exists some $f : X \to \overline{\mathbb{R}}$ such that $F = \operatorname{epi}(f)$.

Proposition 3.2 (Basic properties of the closed epigraph)**.** *Given* $f : X \to \overline{\mathbb{R}}$,

- $X \times \{+\infty\} \subseteq \operatorname{epi}(f)$;
- $(x,z) \in \operatorname{epi}(f)$ *if and only if for every open set* $O \ni x$ *and every* $y > z$, *there exists* $x' \in O$ *such that* $f(x') < y$.

Proof. The first claim follows because $\operatorname{epi}(f)$ contains all (x,z) where $z \geq f(x)$, and $+\infty$ is the maximum of $\overline{\mathbb{R}}$.

For the second claim, first note that a basis of the product topology on $X \times \overline{\mathbb{R}}$ is given by cylinders of the form $O \times I$, where O is an open of X and I is an interval of $\overline{\mathbb{R}}$ that is open in the order topology. From this it follows that $(x,z) \in \operatorname{epi}(f)$ if and only if for every $O \ni x$ and $I \ni z$ we have that $O \times I$ contains some (x',z') with $f(x') \leq z'$. That this implies the second claim is obvious, as for every $y > z$, the set $[-\infty, y)$ is an open interval containing z.

For the reverse implication, there are three possibilities: (a) $+\infty \in I$ (b) $I = [-\infty, y)$ and (c) $I = (w,y)$. The first case is trivial, as we know then $(x,+\infty) \in O \times I$ and $f(x) \leq +\infty$ is always true. The second case is exactly the hypothesis. It suffices to consider only the final case. Let $y > z$ be given and let $x' \in O$ be such that $f(x') < y$. The we know that for every $z' \in [f(x'), y)$ we have $f(x') \leq z'$. It suffices to note that $(w,y) \cap [f(x'), y)$ is non-empty. □

Proposition 3.3. *A closed set* $F \subseteq X \times \overline{\mathbb{R}}$ *is a closed epigraph if and only if both of the following are true:*

1. *for every* $x \in X$, *there exists some* $z \in \overline{\mathbb{R}}$ *such that* $(x,z) \in F$;
2. *if* $(x,z) \in F$ *then* $(x,y) \in F$ *for every* $y \geq z$.

Proof. (*Forward implication*) Assuming F is a closed epigraph, then the first claim follows from the first claim of Prop. 3.2. For the second claim, assume $F = \operatorname{epi}(f)$ for some f. We may assume $y > z$ since when $y = z$ the implication is automatic. By Prop. 3.2 we also know $(x,+\infty) \in F$. So it remains to consider $z < y < +\infty$; in this case y must be real.

By the second claim of Prop. 3.2 it suffices to show that, with $z < y < +\infty$, if for every open set $O \ni x$ and $z' > z$ there is $x' \in O$ such that $f(x') < z'$, then the same holds for z replaced by y. But this follows because if $z' > y$ then $z' > z$ as $y > z$.

(*Reverse implication*) Assuming both conditions, then the fiber $F_x \stackrel{\text{def}}{=} \{z \in \overline{\mathbb{R}} \mid (x,z) \in F\}$ is non-empty and upward closed. As F is closed, this means that F_x is a closed interval $[\inf F_x, +\infty]$. Hence we may define $f(x) = \inf F_x$. By definition $\operatorname{epi}(f) = \operatorname{cl}(F) = F$ as F is closed. □

Definition 3.4 (Lower semi-continuous representative)**.** Following the proof above, given a closed epigraph F, we define $\mathsf{fn}_F : X \to \overline{\mathbb{R}}$ as the function

$$\mathsf{fn}_F(x) = \inf\{z \in \overline{\mathbb{R}} \mid (x,z) \in F\}.$$

It is easy to now check that

Lemma 3.5. *Given a function $f : X \to \overline{\mathbb{R}}$, and set $F = \mathrm{epi}(f)$.*

1. *The function $\mathsf{fn}_F(x) \leq f(x)$ for all x.*
2. *The function fn_F is lower semi-continuous.*
3. *f is lower semi-continuous if and only if $f = \mathsf{fn}_F$.*
4. $\mathrm{epi}(\mathsf{fn}_F) = F$.

Proposition 3.6. *Let F_α be a net of epigraphs in $X \times \mathbb{R}$. Then both $\overline{\tau\text{-lim}}\, F_\alpha$ and $\underline{\tau\text{-lim}}\, F_\alpha$ are epigraphs.*

Proof. We will use Prop. 3.3. We've already established that both $\overline{\tau\text{-lim}}\, F_\alpha$ and $\underline{\tau\text{-lim}}\, F_\alpha$ are closed. And as $X \times \{+\infty\} \subseteq F_\alpha$ for all α by Prop. 3.2, we have $X \times \{+\infty\} \subseteq \overline{\tau\text{-lim}}\, F_\alpha$ (*viz.* $\underline{\tau\text{-lim}}\, F_\alpha$). It suffices to show that each fiber is upward closed.

Under the product topology, given $(x,z) \in \overline{\tau\text{-lim}}\, F_\alpha$ (*viz.* $\underline{\tau\text{-lim}}\, F_\alpha$) if and only if for every open $O \ni x$ and every $\epsilon > 0$, the set $O \times (z-\epsilon, z+\epsilon)$ intersects F_α frequently (*viz.* eventually). Now given $y \geq z$, the super-set $O \times (z-\epsilon, y+\epsilon)$ also intersects F_α frequently (*viz.* eventually). But as F_α is fiberwise upwards closed, if F_α intersects $O \times (z-\epsilon, y+\epsilon)$, it must also intersect $O \times (y-\epsilon, y+\epsilon)$. Together this proves the claim. □

Proposition 3.7. *If a net of functions $f_\alpha : X \to \overline{\mathbb{R}}$ converges pointwise to f, then*

$$\underline{\tau\text{-lim}}\, \mathrm{epi}(f_\alpha) \supseteq \mathrm{epi}(f).$$

Proof. By Prop. 3.2, $(x,z) \in \mathrm{epi}(f)$ if and only if for every open set $O \ni x$ and every $y > z$ there is some $x' \in O$ and $y' \in \mathbb{R}$ such that $f(x') < y' < y$. We may assume without loss of generality that $z < y < +\infty$. Pointwise convergence implies eventually $f_\alpha(x') < y'$. And hence $(x', y') \in \mathrm{epi}(f_\alpha)$ eventually. Hence for any open interval I containing $[z, y')$, we have $(x,z) \in O \times I$ and $O \times I$ intersects $\mathrm{epi}(f_\alpha)$ eventually. This proves $\mathrm{epi}(f) \subseteq \underline{\tau\text{-lim}}\, \mathrm{epi}(f_\alpha)$. □

Proposition 3.8. *If a net of functions $f_\alpha : X \to \overline{\mathbb{R}}$ converges uniformly to f, then*

$$\overline{\tau\text{-lim}}\, \mathrm{epi}(f_\alpha) \subseteq \mathrm{epi}(f).$$

Proof. It suffices to show that if $(x,z) \in \complement\, \mathrm{epi}(f)$, then (x,z) repels $\mathrm{epi}(f_\alpha)$. Our given (x,z) is such that there exists some open $O \ni x$ and $\epsilon > 0$ such that for every $(x', z') \in O \times (z-\epsilon, z+\epsilon)$ we have $f(x') > z'$. This implies $f|_O \geq z + \epsilon$. By uniform convergence we see that the set

$$\{(x', z') \mid z' \geq f(x')\} \cap O \times (z - \epsilon/2, z + \epsilon/2)$$

is eventually empty. Taking the closure of the first factor, and using that the second factor is open, we find that $\mathrm{epi}(f_\alpha) \cap O \times (z - \epsilon/2, z + \epsilon/2)$ is eventually empty, and hence (x,z) repels $\mathrm{epi}(f_\alpha)$. □

Corollary 3.9. *If a net of functions $f_\alpha : X \to \overline{\mathbb{R}}$ converges uniformly to f, then*

$$\tau\text{-lim}\, \mathrm{epi}(f_\alpha) = \mathrm{epi}(f).$$

Proof. Since uniform convergence implies pointwise convergence, the two previous propositions, combined with (2.6), yields

$$\operatorname{epi}(f) \subseteq \underline{\tau\text{-lim}}\operatorname{epi}(f_\alpha) \subseteq \overline{\tau\text{-lim}}\operatorname{epi}(f_\alpha) \subseteq \operatorname{epi}(f).$$

The result follows. □

Definition 3.10. A net of functions $f_\alpha : X \to \overline{\mathbb{R}}$ is said to converge *epigraphically* to $f : X \to \overline{\mathbb{R}}$ if their corresponding closed epigraphs $F_\alpha = \operatorname{epi}(f_\alpha)$ and $F = \operatorname{epi}(f)$ are such that $\tau\text{-lim}\, F_\alpha = F$.

Our Cor. 3.9 shows that uniform convergence of functions implies epigraphical convergence. The reverse implication is not true, and in general epigraphical convergence and pointwise convergence are not logically comparable. We illustrate both of these in the next example.

Example 3.11. Returning to our earlier example, where $K = [-1, 2]$, and our sequence of functions are given by linearly interpolating

$$f_n(-1) = 0 \qquad f_n(-\tfrac{1}{2}) = 1 = f(0) \qquad f_n(\tfrac{1}{n}) = -1 \qquad f_n(\tfrac{2}{n}) = 1 = f_n(2).$$

As discussed, the pointwise limit of these functions is f_∞ given by linearly interpolating

$$f_\infty(-1) = 0 \qquad f_\infty(-\tfrac{1}{2}) = 1 = f_\infty(2).$$

Furthermore this sequence of functions does not converge uniformly, as $\|f_\infty - f_n\|_\infty = 2$.

We will show that the epigraphical limit of f_n exists, but is not equal to f_∞. First note that on the open subset $K_\epsilon = [-1, 0) \cup (\epsilon, 2]$, for $\epsilon \in (0, 1)$, we have that $f_n \to f_\infty$ uniformly (in fact they are eventually equal). So restricted to K_ϵ the epigraphical limit of f_n exists and is equal to f_∞. By Prop. 2.9 this means that

$$(\underline{\tau\text{-lim}}\operatorname{epi}(f_n)) \cap K_\epsilon \times \overline{\mathbb{R}} = (\overline{\tau\text{-lim}}\operatorname{epi}(f_n)) \cap K_\epsilon \times \overline{\mathbb{R}} = \operatorname{epi}(f_\infty) \cap K_\epsilon \times \overline{\mathbb{R}}$$

for every $\epsilon > 0$. It suffices to consider what $\underline{\tau\text{-lim}}\operatorname{epi}(f_n)$ and $\overline{\tau\text{-lim}}\operatorname{epi}(f_n)$ are at $x = 0$.

Consider $(0, z)$ first with $z < -1$. Then the open set $K \times [-\infty, -1)$ is disjoint from $\operatorname{epi}(f_n)$ for any n, and contains $(0, z)$. Hence $(0, z)$ repels $\operatorname{epi}(f_n)$. On the other hand, if $z = -1$, we see that for any open box $(-\epsilon, \epsilon) \times (1 - \delta, 1 + \delta)$, for all sufficiently large n we have $(\frac{1}{n}, -1)$ belongs to the box, and also to $\operatorname{epi}(f_n)$; this shows that $(0, -1)$ belongs to both $\overline{\tau\text{-lim}}\operatorname{epi}(f_n)$ and $\underline{\tau\text{-lim}}\operatorname{epi}(f_n)$.

Observe now that $\operatorname{epi}(f_\infty)$ does not contain $(0, -1)$. This shows that the epigraphical limit of f_n is not f_∞; instead, it is the function

$$f(x) = \begin{cases} f_\infty(x) & x \neq 0 \\ -1 & x = 0 \end{cases}$$

Returning to our motivating question about the values of the infima, we have the following observations. First, notice that for a function $f : X \to \overline{\mathbb{R}}$, we have that

$$\inf f(X) = \inf \mathsf{fn}_{\operatorname{epi}(f)}(X).$$

Now, given a net of epigraphs F_α, and set $\overline{F} = \overline{\tau\text{-lim}}\, F_\alpha$ and $\underline{F} = \underline{\tau\text{-lim}}\, F_\alpha$. Let $z_\alpha = \inf \mathsf{fn}_{F_\alpha}(X)$.

Consider first $z < \liminf z_\alpha$. Then eventually $z_\alpha > z$, and eventually $X \times [-\infty, z) \cap F_\alpha = \emptyset$. This shows that $X \times [-\infty, z) \subseteq \complement(\overline{F})$. This shows then

$$\liminf z_\alpha \leq \inf \mathsf{fn}_{\overline{F}}(X).$$

On the other hand, consider $z > \inf \mathsf{fn}_{\underline{F}}(X)$, so there exists $(x, z') \in \underline{F}$ with $z' < z$; this shows that $X \times [-\infty, z)$ must intersect F_α eventually, and hence z_α must be eventually less than z. Therefore

$$\limsup z_\alpha \leq \inf \mathsf{fn}_{\underline{F}}(X).$$

Let f_α be a net of functions from $X \to \overline{\mathbb{R}}$. Denote by $\overline{f} = \mathsf{fn}_{\overline{\tau\text{-lim}}\,\mathrm{epi}(f_\alpha)}$ and $\underline{f} = \mathsf{fn}_{\underline{\tau\text{-lim}}\,\mathrm{epi}(f_\alpha)}$. The two inequalities derived above can be written as

$$\liminf(\inf f_\alpha) \leq \inf \overline{f} \tag{3.12}$$

$$\limsup(\inf f_\alpha) \leq \inf \underline{f}. \tag{3.13}$$

Prop. 3.7 then implies that if f_α converges to f pointwise, we have

$$\inf \underline{f} \leq \inf f. \tag{3.14}$$

On the other hand, Prop. 3.8 gives that if f_α converges to f uniformly,

$$\inf f \leq \inf \overline{f}. \tag{3.15}$$

And finally from $\mathrm{epi}(\overline{f}) \supseteq \mathrm{epi}(\underline{f})$ we also get $\inf \overline{f} \leq \inf \underline{f}$ so we have that in the case of uniform convergence $\inf f = \lim(\inf f_\alpha)$.

The above discussion also highlights another fact: *if we only know epigraphical convergence of f_α to f, the only useful statement we can prove is*

$$\limsup(\inf f_\alpha) \leq \inf f.$$

We illustrate this with an example.

Example 3.16. Consider the sequence of functions $f_n : \mathbb{R} \to \overline{\mathbb{R}}$, given by

$$f_n(x) = \begin{cases} 0 & x < n \\ -1 & x \geq n \end{cases}$$

It is not too hard to see that f_n converges pointwise and locally uniformly to $f(x) \equiv 0$. And hence $f_n \to 0$ epigraphically. But we have $\inf f_n = -1$ for all n and $\inf f = 0$.

4. Convergence of Infimum

Let f_α be a net of functions from $X \to \overline{\mathbb{R}}$, and denote by $\overline{f} = \mathsf{fn}_{\overline{\tau\text{-lim}}\,\mathrm{epi}(f_\alpha)}$ as before. Our goal is to find sufficient conditions that verifies

$$\liminf(\inf f_\alpha) \overset{?}{\geq} \inf \overline{f}. \tag{4.1}$$

In view of the general inequality (3.12), the above is equivalent to the equality of the two sides. In terms of the corresponding epigraphs F_α and $\overline{F}$, we are equivalently asking: when does the implication

$$F_\alpha \cap X \times [-\infty, z) \text{ is frequently non-empty } \overset{?}{\Longrightarrow} \overline{F} \cap X \times [-\infty, z) \neq \emptyset \tag{4.2}$$

hold. Thinking of $\overline{F}$ as some sort of accumulation points of F_α, we immediately realize that what we need is some sort of compactness condition.

Proposition 4.3. *If X is a topological space and S_α a net of subsets. Let $K \subseteq X$ be compact, be such that $S_\alpha \cap K$ is frequently non-empty. Then $(\overline{\tau\text{-lim}}\, S_\alpha) \cap K$ is non-empty.*

Proof. It is easier to prove the contrapositive. If K is disjoint from $\overline{\tau\text{-lim}}\, S_\alpha$, then every element of K repels S_α, or that for every $x \in K$ there exists $O_x \ni x$ open such that $O_x \cap S_\alpha$ is eventually empty. The set $\{O_x\}$ covers K, so by compactness has a finite subcover labelled $O_1, \dots, O_n$. Corresponding to which are indices $\alpha_1, \dots, \alpha_n$ such that $S_\alpha \cap O_i =$ for all $\alpha \geq \alpha_i$. Finiteness implies therefore $S_\alpha \cap \cup\{O_i\}$ is eventually empty, which implies S_α is eventually disjoint from K. □

Corollary 4.4. *If the epigraphs F_α are such that, for some compact K, and some $z \in \overline{\mathbb{R}}$, we have $F_\alpha \cap (K \times [-\infty, z])$ is frequently non-empty. Then $\overline{F} \cap (K \times [-\infty, z])$ is non-empty.*

Corollary 4.5. *Let $f_\alpha : X \to \overline{\mathbb{R}}$ be a net of functions, and denote by $\overline{f} = \mathsf{fn}_{\overline{\tau\text{-lim}}\,\mathrm{epi}(f_\alpha)}$. Then for any $K \subseteq X$ compact,*

$$\liminf(\inf_K f_\alpha) = \inf_K \overline{f}.$$

Proof. Let $z > \liminf(\inf_K f_\alpha)$, then frequently $\inf_K f_\alpha < z$, or frequently $K \times [-\infty, z] \cap \mathrm{epi}(f_\alpha)$ is non-empty. By the previous corollary we have that $\overline{\tau\text{-lim}}\,\mathrm{epi}(f_\alpha)$ intersects $K \times [-\infty, z]$, and hence there is some $x \in K$ where $\overline{f}(x) \leq z$. This implies $\inf_K \overline{f} \leq z$. As this is true for all $z > \liminf(\inf_K f_\alpha)$, we conclude that $\liminf(\inf_K f_\alpha) \geq \inf_K \overline{f}$. The reverse inequality follows from (3.12). □

This shows that the noncompactness illustrated in Example 3.16 is essentially the only enemy. We can wrap up by tying things back to the notion of epigraphical convergence.

Theorem 4.6. *Fix K a compact topological space. Let $f_\alpha : K \to \overline{\mathbb{R}}$ be a net of functions that converge epigraphically to $f : K \to \overline{\mathbb{R}}$. Then*

(1) *the net $\inf f_\alpha$ converges to $\inf f$.*
(2) *let $S_\alpha \subseteq K$ be given by $S_\alpha = f_\alpha^{-1}(\inf f_\alpha)$. If f is additionally lower semi-continuous, then $\overline{\tau\text{-lim}}\, S_\alpha \subseteq f^{-1}(\inf f)$.*

Proof.
(1) By Cor. 4.5 combined with (3.13) we have

$$\inf f \leq \liminf(\inf f_\alpha) \leq \limsup(\inf f_\alpha) \leq \inf f.$$

(2) If $x \in \overline{\tau\text{-lim}}\, S_\alpha$, then for every open set $O \ni x$ we know S_α frequently intersects O. By definition $f_\alpha|_{S_\alpha} = \inf f_\alpha$, which converges to $\inf f$ by the first part. So for every open interval I containing $\inf f$, we have S_α is frequently in O and $f_\alpha(S_\alpha)$ is eventually in I, which means that the set $S_\alpha \times \{\inf f_\alpha\}$ frequently intersects $O \times I$. Thus $(x, \inf f) \in \overline{\tau\text{-lim}}\,\mathrm{epi}(f_\alpha)$. Since we assumed epigraphical convergence, $\overline{\tau\text{-lim}}\,\mathrm{epi}(f_\alpha) = \mathrm{epi}(f)$, and hence we have $(x, \inf f) \in \mathrm{epi}(f)$. Using that f is assumed to be lower semi-continuous, this means $f(x) \leq \inf f$. By the definition of infimum this is in fact an equality.

□

The second claim of the theorem can be regarded as the *upper semi-continuity* of the minima. More precisely, let (X, τ) and (Y, σ) be topological spaces. We can consider a function $f : Y \to 2^X$ that outputs subsets of X. Given $y \in Y$, we can take the canonical net γ that converges to y, and consider $\overline{\tau\text{-lim}}\, f \circ \gamma$ and $\underline{\tau\text{-lim}}\, f \circ \gamma$.

Our previous result suggests that these two sets can be considered as a sort of limit superior and inferior respectively, as we know $\underline{\tau\text{-lim}}\, f \circ \gamma \subseteq \overline{\tau\text{-lim}}\, f \circ \gamma$, so that they have the correct relation with respect to the set inclusion ordering on 2^X. Then we may say that f is continuous at y if $\underline{\tau\text{-lim}}\, f \circ \gamma = \overline{\tau\text{-lim}}\, f \circ \gamma$. Similarly, upper semi-continuity can be defined to be

$$\overline{\tau\text{-lim}}\, f \circ \gamma \subseteq f(y). \tag{4.7}$$

With these definitions, the above Theorem can be rephrased as follows.

Theorem 4.8. *Fix K a compact topological space and X a topological space. Consider a mapping $f : K \times X \to \overline{\mathbb{R}}$. Fix x_0 a point in X, and suppose as a family of $K \to \overline{\mathbb{R}}$ functions $f(\cdot, x)$ converges to $f(\cdot, x_0)$ epigraphically as $x \to x_0$. Furthermore suppose $f(\cdot, x_0)$ is lower semi-continuous. Then*

(1) *The function $g : X \to \overline{\mathbb{R}}$ given by $g(x) = \inf_{k \in K} f(k, x)$ is continuous at x_0.*
(2) *The set-valued function $h : X \to 2^K$ given by $h(x) = \{k \in K \mid f(k, x) = g(x)\}$ is upper semi-continuous (in the sense discussed in the previous paragraph) at x_0.*

For applications, we would typically want to assume that f is a continuous mapping. Then the compactness of K implies that for every $\epsilon > 0$ and for every x_0 there exists an open set $O \ni x_0$ such that for $x \in O$ and $k \in K$, we have $|f(k, x) - f(k, x_0)| < \epsilon$. In other words, this ensures $f(\cdot, x)$ converges to $f(\cdot, x_0)$ uniformly as $x \to x_0$, and hence the convergence is also epigraphical. The hypothesis also implies that $f(\cdot, x_0)$ is lower semi-continuous, so all hypotheses are met. We summarize this application as a corollary.

Corollary 4.9. *Let K be a compact topological space and X a topological space. Take $f : K \times X \to \overline{\mathbb{R}}$ a continuous mapping and define $g : X \to \overline{\mathbb{R}}$ by $g(x) = \inf_{k \in K} f(k, x)$, and $h : X \to 2^K$ by $h(x) = \{k \in K \mid f(k, x) = g(x)\}$. Then:*

- *g is continuous.*
- *h is upper semi-continuous.*
- *$h(x)$ is not empty for any x.*

If furthermore $h(x)$ is known to be a singleton set for all x, then $h(x) = \{\tilde{h}(x)\}$ where $\tilde{h} : X \to K$ is continuous.

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