ORDER CONVERGENCE ON COMPLETE LATTICES

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ABSTRACT. We explore some features of order convergence on a complete lattice, focusing specifically on where the general case agrees and differs from the elementary setting where the lattice is the extended real line $\overline{\mathbb{R}}$. Using this language, we revisit both classical concepts such as convergences of nets in topological spaces and pointwise convergence of functions, as well as more modern ideas in optimization related to the convergence of nets of sets and epigraphical convergence of functions.

1. REVIEW OF TERMINOLOGY

The material in this section is standard. Convenient references include [1–3, 5, 8].

1.1. **Posets and Complete Lattices.** A partially ordered set (or *poset*) is a set P equipped with a relation $\leq$ that is reflexive, antisymmetric, and transitive. In a poset P, given $x \in P$, we will denote by

$$\begin{aligned} \uparrow(x) &\stackrel{\text{def}}{=} \{y \in P \mid x \leq y\}; \\ \downarrow(x) &\stackrel{\text{def}}{=} \{z \in p \mid z \leq x\}. \end{aligned} \tag{1.1}$$

Similarly, given a subset $S \subseteq P$, we denote by

$$\uparrow(S) \stackrel{\text{def}}{=} \bigcup\{\uparrow s \mid s \in S\}, \qquad \downarrow(S) \stackrel{\text{def}}{=} \bigcup\{\downarrow(s) \mid s \in S\}. \tag{1.2}$$

If we take the intersection instead, we define the sets of *upper bounds* and *lower bounds*.

$$\mathsf{ub}(S) \stackrel{\text{def}}{=} \bigcap\{\uparrow(s) \mid s \in S\}, \qquad \mathsf{lb}(S) \stackrel{\text{def}}{=} \bigcap\{\downarrow(s) \mid s \in S\}. \tag{1.3}$$

A set $A \subseteq P$ is said to be an *upper set* (resp. *lower set*) if $a \in A \implies \uparrow(a) \subseteq A$ (resp. $\downarrow(a) \subseteq A$). It is easily checked that both $\uparrow(S)$ and $\mathsf{ub}(S)$ are upper sets.

Given a subset S, an element $s \in S$ is *maximal* if $\uparrow(s) \cap S = s$; *minimality* is similarly defined. An element $s \in S$ is the (unique) *maximum element* if $S \subseteq \downarrow(s)$. Similarly we can define the *minimum element*. Most sets in most posets do not contain maximum or minimum elements.

Definition 1.4 (Complete Lattice)**.** A poset $(L, \leq)$ is said to be a *complete lattice* if, for every $S \subseteq L$,

- $\mathsf{ub}(S)$ has a minimum element, and
- $\mathsf{lb}(S)$ has a maximum element.

Date: March 4, 2025.

In a complete lattice, we denote by

$$\sup(S) \stackrel{\text{def}}{=} \text{the minimum element of } \mathsf{ub}(S),$$

$$\inf(S) \stackrel{\text{def}}{=} \text{the maximum element of } \mathsf{lb}(S).$$

Thus, we use (deliberately) the same notation as in real variables where sup denotes the least upper bound and inf denotes the greatest lower bound. It is clear that if S has a maximum element then it would be $\sup(S)$, and if S has a minimum element then it would be $\inf(S)$; the reverse implication fails as usual.

As this note is intended to be expository, for readers familiar with the order structure of the real numbers, a prototypical example of a complete lattice is *not* the real numbers, but the extended reals $\overline{\mathbb{R}} = [-\infty, +\infty]$. Dedekind completeness of the ordinary real numbers almost gives us the existence of sup and inf for all sets; we only impose the condition that the sets are *a priori* bounded. The additional elements $\pm\infty$ are included to formally give meaning to sup and inf of unbounded sets.

1.2. **Complete Distributivity.** For handling expressions involving nested suprema and infima, it is sometimes convenient to know how the two interact. We formalize this as a definition.

Definition 1.5 (Complete distributivity)**.** A complete lattice $(L, \leq)$ is said to be *completely distributive* if for every pair of index sets I, J and every mapping $x : I \times J \to L$, we have

$$\sup_{i\in I} \inf_{j\in J} x(i,j) = \inf_{f\in J^I} \sup_{i\in I} x(i, f(i)).$$

It is known that being completely distributive is self-dual, so that a complete lattice is completely distributive if and only if the definition also holds with sup and inf swapped.

Examining the defining condition, we see that for any $f \in J^I$, we have

$$x(i, f(i)) \geq \inf_{j\in J} x(i,j).$$

And hence

$$\sup_{i\in I} x(i, f(i)) \geq \sup_{i\in I} \inf_{j\in J} x(i,j).$$

As this holds for any choice of f, we find that

Proposition 1.6. *For any complete lattice, distributive or otherwise,*

$$\sup_{i\in I} \inf_{j\in J} x(i,j) \leq \inf_{f\in J^I} \sup_{i\in I} x(i, f(i))$$

holds for any $x : I \times J \to L$. Similarly, we also have

$$\inf_{i\in I} \sup_{j\in J} x(i,j) \geq \sup_{f\in J^I} \inf_{i\in I} x(i, f(i)).$$

Thus we see that one inequality in the definition of complete distributivity is universal, and the content of the definition lies in the other inequality.

Example 1.7. The basic example of a completely distributive complete lattice is the extended real line. In this case the key observation is that by the fact that the ordering on the extended real line is a total order, if $y > \sup_{i\in I} \inf_{j\in J} x(i,j)$, then there exists a function f such that $x(i,f(i)) \leq y$. This shows that $\sup_{i\in I} x(i,f(i)) \leq y$. And hence

$$y > \sup_{i\in I}\inf_{j\in J} x(i,j) \implies y \geq \inf_{f\in J^I}\sup_{i\in I} x(i,f(i)).$$

This shows the desired equality.

Example 1.8. Let L be the power set of some set X, and $\leq$ be given by subset inclusion. Then this forms a completely distributive complete lattice. Here the supremum over a family of sets is their union, and infimum is their overall intersection.

To prove this, it is enough to show that

$$z \notin \sup_{i\in I}\inf_{j\in J} x(i,j) \implies z \notin \inf_{f\in J^I}\sup_{i\in I} x(i,f(i)).$$

In our setting, the hypothesis is equivalent to $z \notin \inf_{j\in J} x(i,j)$ for any i, which implies for every i there is some j such that $z \notin x(i,j)$. Hence there exists[1] a function $f : I \to J$ such that $z \notin x(i,f(i))$ for any i, which implies the desired conclusion.

Example 1.9. For an example of a complete lattice that is not completely distributive, Let L be the set of all closed subsets of $\mathbb{R}$ organized by inclusion. If S is a collection of closed sets, then

$$\inf S = \bigcap S, \qquad \sup S = \mathrm{cl}(\bigcup S).$$

(We use $\mathrm{cl}(\cdot)$ the topological closure.) This shows that L is a complete lattice.

To show that L is not completely distributive, let $I = J = \mathbb{N}$ and $x(i,j)$ be the singleton sets

$$x(i,j) = \Big\{\frac{1}{\max(i,j)}\Big\}.$$

Fixing i, we see that $\inf_{j\in J} x(i,j) = \emptyset$. On the other hand, for any $f : \mathbb{N} \to \mathbb{N}$ we have that $x(i,f(i)) \subseteq (0, \frac{1}{i}]$ and hence as a sequence of points (indexed by i) converges to 0. Hence $0 \in \sup_{i\in I} x(i,f(i))$ for any f. Thus we see that

$$\emptyset = \sup_{i\in I}\inf_{j\in J} x(i,j) \quad \text{while} \quad 0 \in \inf_{f\in J^I}\sup_{i\in I} x(i,f(i))$$

showing that L is not completely distributive.

1.3. **Nets.** A *directed set* $(A,\leq)$ is a set equipped with a relation $\leq$ that is reflexive and transitive (but not necessarily antisymmetric), such that for any *finite subset* $S \subseteq A$ the set $\mathsf{ub}(S)$ is non-empty[2].

Given X a set, and $(A,\leq)$ a directed set. A *net* (or *Moore-Smith sequence*) in X, indexed by $(A,\leq)$, is an assignment $x_\alpha \in X$ to each element $\alpha \in A$. A common example is when we use $(\mathbb{N},\leq)$ with the standard ordering as the underlying directed set. In this case our net is what classically we call "sequences". When referring to the entire net (as opposed to a single item in the net), we will enclose it in parentheses as (x_α).

For convenience, we will denote by

$$x_{\uparrow(\alpha)} \stackrel{\text{def}}{=} \{x_\beta \mid \beta \in \uparrow(\alpha)\}. \tag{1.10}$$

Sets of these form are called *residual sets* of the net (x_α).

[1]We accept axiom of choice.

[2]While we defined $\mathsf{ub}(S)$ for subsets of a poset, the same words and symbols can be used for any set equipped with any relation.

Given a net (x_α), we say that a mathematical statement is *eventually true* for (x_α) if it is true for some residual set $x_{\uparrow(\alpha_0)}$. On the other hand, a mathematical statement is *frequently true* if, for every $\alpha \in A$, there is $\beta \geq \alpha$ such that the statement is true for x_β.

2. Order Convergences

Fundamental to topology is the determination whether a given net converges. In this section we will describe some notions of convergence for nets in a complete lattice. There are many order topologies considered on various forms of posets, some of which can be found in [5, 6, 9]. In the MSC2020 database [4] the study of topological lattices and continuous lattices are given their own subject codes (06B30 and 06B35 respetively). Our starting point is analytic in nature, and we draw analogies from real-variables theory.

The classical definition of limits superior and inferior can be reproduced for nets in a complete lattice. We have:

Definition 2.1. Let $(L, \leq)$ be a complete lattice, and (x_α) a net in L indexed by A, we can define its limits superior and inferior using via

$$\limsup(x_\alpha) = \inf\{\sup(x_{\uparrow(\beta)}) \mid \beta \in A\}$$
$$\liminf(x_\alpha) = \sup\{\inf(x_{\downarrow(\beta)}) \mid \beta \in A\}.$$

2.1. Eventual and frequent bounds.

Given a net in a complete lattice, we can talk about whether an element is eventually or frequently an upper (lower) bound of the net. We formalize this definition:

Definition 2.2 (Eventual and frequent upper/lower bounds)**.** Let $(L, \leq)$ be a complete lattice, and (x_α) a net in L indexed by A. The set of *eventual upper bounds* of (x_α) is defined to be

$$^{(\mathsf{e})}\mathsf{ub}(x_\alpha) \stackrel{\text{def}}{=} \{y \in L \mid y \geq x_\alpha \text{ is eventually true }\}.$$

The set of *frequent upper bounds* of (x_α) is defined to be

$$^{(\mathsf{f})}\mathsf{ub}(x_\alpha) \stackrel{\text{def}}{=} \{y \in L \mid y \geq x_\alpha \text{ is frequently true }\}.$$

We similarly define *eventual lower bounds* and *frequent lower bounds*, and denote them by $^{(\mathsf{e})}\mathsf{lb}(x_\alpha)$ and $^{(\mathsf{f})}\mathsf{lb}(x_\alpha)$ respectively.

Their corresponding infima and suprema can be interpreted as version of the limits superior and limits inferior.

Definition 2.3. Let $(L, \leq)$ be a complete lattice, and (x_α) a net in L indexed by A. We define

$$^{(\mathsf{e})}\limsup(x_\alpha) \stackrel{\text{def}}{=} \inf {}^{(\mathsf{e})}\mathsf{ub}(x_\alpha)$$
$$^{(\mathsf{f})}\limsup(x_\alpha) \stackrel{\text{def}}{=} \inf {}^{(\mathsf{f})}\mathsf{ub}(x_\alpha)$$
$$^{(\mathsf{e})}\liminf(x_\alpha) \stackrel{\text{def}}{=} \sup {}^{(\mathsf{e})}\mathsf{lb}(x_\alpha)$$
$$^{(\mathsf{f})}\liminf(x_\alpha) \stackrel{\text{def}}{=} \sup {}^{(\mathsf{f})}\mathsf{lb}(x_\alpha)$$

The eventual versions turns out to be the classical version of limit superior and limit inferior in a complete lattice.

Proposition 2.4. *Given a net (x_α) in a complete lattice, we have the equalities*

$$^{(\mathsf{e})}\limsup(x_\alpha) = \limsup(x_\alpha) \quad \textit{and} \quad ^{(\mathsf{e})}\liminf(x_\alpha) = \liminf(x_\alpha).$$

Proof. We prove the first formula, the second is identical.

First, notice that if $y \in {}^{(\mathsf{e})}\mathsf{ub}(x_\alpha)$, then it is an upper bound of $x_{\uparrow(\alpha_0)}$ for some $\alpha_0 \in A$, which means that $y \geq \sup(x_{\uparrow(\alpha_0)})$. This means that

$$\mathsf{lb}(\{\sup(x_{\uparrow(\beta)}) \mid \beta \in A\}) \subseteq \mathsf{lb}({}^{(\mathsf{e})}\mathsf{ub}(x_\alpha)).$$

On the other hand, by definition for any $\beta \in A$ we have $\sup(x_{\uparrow(\beta)})$ is an eventual upper bound, and hence

$$\mathsf{lb}(\{\sup(x_{\uparrow(\beta)}) \mid \beta \in A\}) \supseteq \mathsf{lb}({}^{(\mathsf{e})}\mathsf{ub}(x_\alpha)).$$

So the two sets of lower bounds agree, and so the infima must agree. □

The definitions immediately imply the following facts:

- ${}^{(\mathsf{e})}\mathsf{ub}(x_\alpha) \subseteq {}^{(\mathsf{f})}\mathsf{ub}(x_\alpha)$;
- ${}^{(\mathsf{e})}\mathsf{lb}(x_\alpha) \subseteq {}^{(\mathsf{f})}\mathsf{lb}(x_\alpha)$;
- $y \in {}^{(\mathsf{e})}\mathsf{lb}(x_\alpha)$ and $z \in {}^{(\mathsf{f})}\mathsf{ub}(x_\alpha)$ implies $y \leq z$.
- $y \in {}^{(\mathsf{f})}\mathsf{lb}(x_\alpha)$ and $z \in {}^{(\mathsf{e})}\mathsf{ub}(x_\alpha)$ implies $y \leq z$.

Note however in general frequent lower bounds and frequent upper bounds may fail to be comparable. A consequence of these comparison relations is that we have

$$^{(\mathsf{e})}\liminf(x_\alpha) \leq \begin{matrix} ^{(\mathsf{f})}\liminf(x_\alpha) \\ ^{(\mathsf{f})}\limsup(x_\alpha) \end{matrix} \leq {}^{(\mathsf{e})}\limsup(x_\alpha). \tag{2.5}$$

Proposition 2.6. *If $(L, \leq)$ is a completely distributive complete lattice, then*

$$^{(\mathsf{e})}\liminf(x_\alpha) = {}^{(\mathsf{f})}\limsup(x_\alpha)$$
$$^{(\mathsf{f})}\liminf(x_\alpha) = {}^{(\mathsf{e})}\limsup(x_\alpha).$$

Proof. We prove the first claim, the second is similar. Note that one direction is already proven as part of (2.5); it suffice to prove the reverse.

Given our net x_α, define $\xi : A \times A \to L$ by

$$\xi(\alpha, \beta) = \begin{cases} x_\beta & \alpha \leq \beta \\ x_\alpha & \text{otherwise} \end{cases}.$$

Then for $\alpha \in A$, we have that $\{x(\alpha, \beta) \mid \beta \in A\} = x_{\uparrow(\alpha)}$. So by Prop. 2.4 we have that

$$^{(\mathsf{e})}\liminf(x_\alpha) = \sup_{\alpha \in A} \inf_{\beta \in A} \xi(\alpha, \beta).$$

Since the lattice is assumed to be completely distributive, we have also

$$^{(\mathsf{e})}\liminf(x_\alpha) = \inf_{f \in A^A} \sup_{\alpha \in A} \xi(\alpha, f(\alpha)).$$

Now, suppose that $f \in A^A$. Let $g : A \to A$ be given by

$$g(\alpha) = \begin{cases} f(\alpha) & \alpha \leq f(\alpha) \\ \alpha & \text{otherwise} \end{cases}.$$

Then we have that $\xi(\alpha, f(\alpha)) = \xi(\alpha, g(\alpha)) = x_{g(\alpha)}$, and that $g(\alpha) \in \uparrow(\alpha)$ for every α. Therefore $\sup_{\alpha \in A} \xi(\alpha, g(\alpha)) \geq x_{g(\alpha)}$ for each α, which shows that $\sup_{\alpha \in A} \xi(\alpha, g(\alpha)) \in$

${}^{(\mathsf{f})}\mathsf{ub}(x_\alpha)$. Taking the infima we see therefore ${}^{(\mathsf{e})}\liminf(x_\alpha) \geq {}^{(\mathsf{f})}\limsup(x_\alpha)$ as needed. □

Example 2.7. We illustrate the distinction between ${}^{(\mathsf{e})}\liminf(x_\alpha)$ and ${}^{(\mathsf{f})}\limsup(x_\alpha)$ using the non-completely-distributive lattice L, consisting of closed subsets of a topological space X. This example is closely related to Ex. 1.9.

Let X be a topological space. Let L be the collection of all closed subsets of X. Consider p_α a net in X, and set $x_\alpha = \{p_\alpha\}$ the corresponding net of singleton sets.

We have that $y \in {}^{(\mathsf{e})}\liminf(x_\alpha)$ if and only if p_α is eventually constant (and hence equal to y), and $y \in {}^{(\mathsf{f})}\liminf(x_\alpha)$ if and only if there is a subnet of p_α that is eventually constant and equals to y.

The set ${}^{(\mathsf{e})}\limsup(x_\alpha)$ can be easily seen to be the set of all accumulation points of (x_α). Finally, let's consider ${}^{(\mathsf{f})}\limsup(x_\alpha)$. By definition $q \in {}^{(\mathsf{f})}\limsup(x_\alpha)$ if and only if $q \in y$ for every $y \in {}^{(\mathsf{f})}\mathsf{ub}(x_\alpha)$. In other words, $q' \notin {}^{(\mathsf{f})}\limsup(x_\alpha)$ if and only if there is some $y \in {}^{(\mathsf{f})}\mathsf{ub}(x_\alpha)$ such that $q' \notin y$. This is equivalent to saying that there exists an open neighorhood $(X \setminus y)$ of q' such that x_α is frequently outside of said neighborhood. Which is equivalent to the statement that (x_α) does not converge to q'. Therefore, we in fact have the following characterization: ${}^{(\mathsf{f})}\limsup(x_\alpha)$ is the set $\{q \in X \mid x_\alpha \to q\}$.

Consider now the following explicit examples.

- Let X be the non-Hausdorff topological space of the real line with two zeros. Let p_n be the sequence with $p_{2k} = 0_2$ (the second copy of 0) and $p_{2k-1} = \frac{1}{k}$. Then we have

$$ {}^{(\mathsf{e})}\liminf(x_n) = \emptyset \subsetneq {}^{(\mathsf{f})}\liminf(x_n) = \{0_2\} \subsetneq {}^{(\mathsf{f})}\limsup(x_n) = \{0_1, 0_2\} = {}^{(\mathsf{e})}\limsup(x_n). $$

- Let X be the standard real line, and p_n given by $p_{2k} = 1$ and $p_{2k-1} = \frac{1}{k}$. Then we have

$$ {}^{(\mathsf{e})}\liminf(x_n) = \emptyset = {}^{(\mathsf{f})}\limsup(x_n) \subsetneq {}^{(\mathsf{f})}\liminf(x_n) = \{0\} \subsetneq {}^{(\mathsf{e})}\limsup(x_n) = \{0, 1\}. $$

Example 2.8 (Inner and outer limits). Let X be a topological space, and S_α be a net of closed[3] subsets thereof. The *inner and outer limits* of set-valued analysis (see [7, Ch. 4]) are defined as follows:

- The *outer limit* of S_α consists of all points $x \in X$ with the property "every open neighborhood $O \ni x$ intersects S_α frequently."
- The *inner limit* of S_α consists of all points $x \in X$ with the property "every open neighborhood $O \ni x$ intersects S_α eventually."

These two notions are in fact related to our definitions above.

Observe that the statement "x is not in the outer (inner) limit" is equivalent to the statement that "there is an open neighborhood $O \ni x$ that is eventually (frequently) disjoint from S_α." This shows that both the outer and inner limits are closed sets. Next, our statement is equivalent to "there is a closed set C disjoint from x that eventually (frequently) contains S_α." This we rephrase as "x is disjoint from some eventual (frequent) upper bound of S_α." Therefore we conclude:

[3]In the literature the inner and outer limits are usually defined for arbitrary sets. But since given an open set O, we have that O intersects a set S if and only if O intersects its closure, we see that the inner/outer limits of S_α is invariant if we replace each element of the net by its closure. From the point of view of lattice theory it however makes more sense to think only about closed sets.

- The outer limit of S_α is exactly ${}^{(e)}\limsup(S_\alpha)$, where we use as our lattice the set of all closed subsets of X;
- and similarly the inner limit of S_α is exactly ${}^{(f)}\limsup(S_\alpha)$.

2.2. **Convergence Concepts.** Starting from (2.5), one is led to define convergences through the coincidence of two or more of the limiting quantities. The most basic is the notion with which we defined convergence on the (extended) real line:

Definition 2.9 (Moore–Smith Order Covergence)**.** Let $(L,\leq)$ be a complete lattice. We say that a net x_α converges in the Moore–Smith sense if ${}^{(e)}\limsup(x_\alpha) = {}^{(e)}\liminf(x_\alpha)$. (By (2.5) we must also have that ${}^{(f)}\limsup(x_\alpha)$ and ${}^{(f)}\liminf(x_\alpha)$ equal the same value.)

An alternative notion is formed when we instead compare eventual upper bounds to frequent upper bounds; that this is a useful notion in topology follows from the analyses in Examples 2.7 and 2.8.

Definition 2.10 (Painlevé–Kuratowski Convergence)**.** Let $(L,\leq)$ be a complete lattice. We say that a net x_α converges in the Painlevé–Kuratowski sense if ${}^{(e)}\limsup(x_\alpha) = {}^{(f)}\limsup(x_\alpha)$. (Note that this says nothing about ${}^{(e)}\liminf(x_\alpha)$ or ${}^{(f)}\liminf(x_\alpha)$.)

From (2.5) and Prop. 2.6, we conclude the following:

Proposition 2.11.

(1) *Moore–Smith convergence implies Painlevé–Kuratowski convergence.*
(2) *On completely distributive complete lattices, Moore–Smith and Painlevé–Kuratowski convergences coincide.*

Remark 2.12. One may of course ask whether it is meaningful to also consider the coincidence of ${}^{(e)}\limsup(x_\alpha)$ with ${}^{(f)}\liminf(x_\alpha)$. That this concept is less useful comes again from Prop. 2.6, which asserts that the two quantities coincide for any net in any completely distributive lattice. Hence it seems less likely that the coincidence of these two quantities will provide a meaningful/nontrivial notion of convergence in general.

Concepts familiar from the Moore–Smith convergence on the extended real line can be carried over in general to Painlevé–Kuratowski convergence also.

Theorem 2.13 (Monotone convergence)**.** *If x_α is a monotone net in a complete lattice, then it converges in the Painlevé–Kuratowski sense.*

Proof. If x_α is decreasing, then every frequent upper bound is clearly an eventual upper bound, and the result follows trivially.

Suppose x_α is increasing, and y a frequent upper bound. Then for every $\beta \in A$ there exists $\gamma \geq \beta$ such that $y \geq x_\gamma$. But since x is assumed to be increasing, we have $x_\gamma \geq x_\beta$, showing that $y \geq x_\beta$. Since β is arbitrary we conclude that y is necessarily an eventual upper bound. And the result follows. □

Theorem 2.14 (Squeeze theorem)**.** *Let A be a fixed index set and $x_\alpha, y_\alpha, z_\alpha$ be three nets in a complete lattice, such that for every α we have $x_\alpha \leq y_\alpha \leq z_\alpha$. If x_α and z_α both Painlevé–Kuratowski converges to the same limit, then so does y_α.*

Proof. The ordering implies

$$ {}^{(e)}\mathsf{ub}(z_\alpha) \subseteq {}^{(e)}\mathsf{ub}(y_\alpha) \subseteq {}^{(e)}\mathsf{ub}(x_\alpha) $$

and

$$^{(\mathsf{f})}\mathsf{ub}(z_\alpha) \subseteq {}^{(\mathsf{f})}\mathsf{ub}(y_\alpha) \subseteq {}^{(\mathsf{f})}\mathsf{ub}(x_\alpha).$$

Taking the infima we find

$$^{(\mathsf{e})}\limsup(z_\alpha) \geq {}^{(\mathsf{e})}\limsup(y_\alpha) \geq {}^{(\mathsf{e})}\limsup(x_\alpha)$$

and

$$^{(\mathsf{f})}\limsup(z_\alpha) \geq {}^{(\mathsf{f})}\limsup(y_\alpha) \geq {}^{(\mathsf{f})}\limsup(x_\alpha).$$

The result follows. □

3. Function convergences

Let X be a topological space, and $f : X \to \overline{\mathbb{R}}$ a function. By the *epigraph* of f we refer to the subset

$$\operatorname{epi}(f) \stackrel{\text{def}}{=} \{(x,s) \in X \times \overline{\mathbb{R}} \mid f(x) \leq s\}. \tag{3.1}$$

We say that f is *lower semi-continuous* if $\operatorname{epi}(f)$ is closed in the product topology.

We can partially order functions using set inclusion of the epigraphs:

$$f \preceq g \iff \operatorname{epi}(f) \subseteq \operatorname{epi}(g) \iff \forall x \in X, f(x) \geq g(x). \tag{3.2}$$

Note that this implies the *opposite* comparison of the pointwise values of the functions in question.

Observe that if $\mathcal{F}$ is a family of functions $X \to \overline{\mathbb{R}}$, if we define

$$g(x) \stackrel{\text{def}}{=} \sup\{f(x) \mid f \in \mathcal{F}\} \tag{3.3}$$

to be the pointwise supremum, we also have that

$$\operatorname{epi}(g) = \bigcap\{\operatorname{epi}(f) \mid f \in \mathcal{F}\}.$$

3.1. **Pointwise convergence.** First I claim that if I set L to be the set of all $X \to \overline{\mathbb{R}}$ functions, then it is a complete lattice with order given by (3.2). As usual it suffices to show that L is inf-complete. But it is not too hard to check that in our ordering, the infimum of a family $\mathcal{F}$ is precisely the point-wise supremum as given in (3.3).

The following claims we left as exercise to the reader:

Proposition 3.4. *Let L be the complete lattice of $X \to \overline{\mathbb{R}}$ functions with the ordering* (3.2).

(1) *L is completely distributive.*
(2) *Moore–Smith convergence on L is equivalent to Painlevé convergence on L, and they are both equivalent to pointwise convergence of the net of functions.*

Note that in this setting we made no use of the topology of X; for all intents and purposes X can be just any set with or without topology.

3.2. **Γ-convergence.** Now instead we let L be the set of all lower semi-continuous functions from $X \to \overline{\mathbb{R}}$. We still use the same ordering (3.2).

I claim that this still makes L a complete lattice. Again it suffices to show that L is inf-complete. Given a family $\mathcal{F}$, we again set g to be the pointwise supremum. Our arguments before indicated that g has epigraph equal to $\bigcap\{\operatorname{epi}(f) \mid f \in \mathcal{F}\}$. But as each $f \in \mathcal{F}$ is lower semi-continuous, their epigraphs are closed and hence their corresponding intersections are also closed. This shows that g is also an element of L.

Definition 3.5. By Γ-*convergence* or *epigraphical convergence* we refer to convergence in the complete lattice of lower semi-continuous functions in the Painlevé-Kuratowski sense.

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